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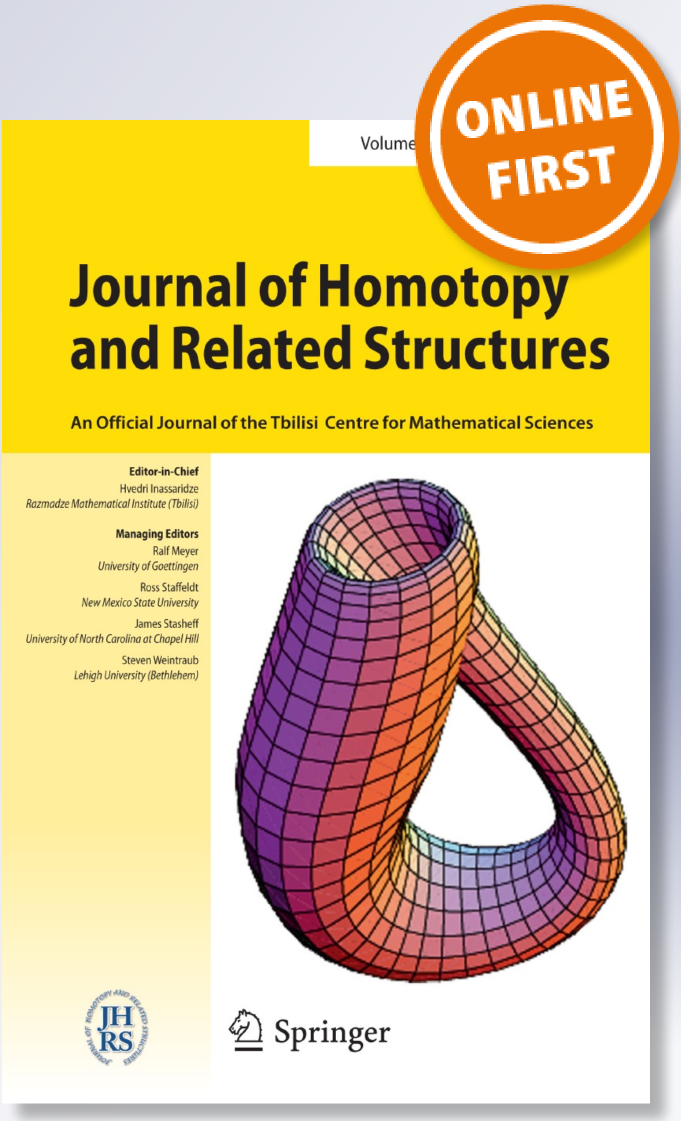

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Abstract We show that the algebraic K-theory of generalized archimedean valuation rings occurring in Durov's compactification of the spectrum of a number ring is given by stable homotopy groups of certain classifying spaces. We also show that the "residue field at infinity" is badly behaved from a K-theoretic point of view.

Keywords Algebraic K-theory · Complexes of groups · Infinite place

1 Introduction

In number theory, it is a universal principle that the spectrum of $\mathbb{Z}$ should be completed with an infinite prime. This is corroborated, for example, by Ostrowski's theorem, the product formula

$$\prod_{p \leq \infty} |x|_p = 1, \quad x \in \mathbb{Q}^\times,$$

the Hasse principle, Artin–Verdier duality, and functional equations of L-functions.

This "compactification" $\operatorname{Spec} \widehat{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z} \cup \{\infty\}$ was just a philosophical device until recently: Durov has proposed a rigorous framework which allows for a discussion of, say, $\mathbb{Z}_{(\infty)}$, the local ring of $\operatorname{Spec} \widehat{\mathbb{Z}}$ at $p = \infty$ [1]. The purpose of this work is to study the K-theory of the so-called generalized rings intervening at the infinite place.

Algebraic K-theory is a well-established, if difficult, invariant of arithmetical schemes. For example, the pole orders of the Dedekind ζ-function $\zeta_F(s)$ of a number

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field F are expressible by the ranks of the K-theory groups of $\mathcal{O}_F$, the ring of integers. By definition, K-theory only depends on the category of projective modules over a ring. Therefore, this interacts nicely with Durov's theory of *generalized rings* which describes (actually: defines) such a ring R by defining its free modules. For example, the free $\mathbb{Z}_{(\infty)}$-module of rank n is defined as the n-dimensional octahedron, i.e.,

$$\mathbb{Z}_{(\infty)}(n) := \left\{ (x_1, \ldots, x_n) \in \mathbb{Q}^n, \sum_i |x_i| \leq 1 \right\}.$$

The abstract theory of such modules is a priori more complicated than in the classical case since $\mathbb{Z}_{(\infty)}$-modules fail to build an abelian category. Nonetheless, using Waldhausen's $S_\bullet$-construction it is possible to study the *algebraic K-theory* of $\mathbb{Z}_{(\infty)}$ and similar rings occurring for other number fields (Theorem 3.10, Definition 3.12).

Theorem 3.14. *The K-groups of $\mathbb{Z}_{(\infty)}$ are given by*

$$K_i(\mathbb{Z}_{(\infty)}) = \pi_i^{\mathrm{s}}(B\mu_2 \sqcup \{*\}, *) = \begin{cases} \mathbb{Z} & i = 0 \ (Durov[Dur, 10.4.19]) \\ \mathbb{Z}/2 \oplus \mu_2 & i = 1 \\ \text{a finite group} & i > 1. \end{cases}$$

The $\mathbb{Z}/2$-part in K_1 stems from the first stable homotopy group π_1^s, while $\mu_2 = \{\pm 1\}$ arises as the subgroup of $\mathbb{Z}_{(\infty)}$ of elements of norm 1, i.e., the subgroup of (multiplicative) units of $\mathbb{Z}_{(\infty)}$. The finite K-group for $i > 1$ is the abutment of an Atiyah–Hirzebruch spectral sequence.

This theorem is proven for more general generalized valuation rings including $\mathcal{O}_{F(\sigma)}$, the ring corresponding to an infinite place σ of a number field F. In this case the group μ_2 above is replaced by the group $\{x \in F, |\sigma(x)| = 1\}$. The basic point is this: the only admissible monomorphisms (i.e., the ones occurring in the $S_\bullet$-construction of K-theory)

$$\mathbb{Z}_{(\infty)}(1) = [-1, 1] \cap \mathbb{Q} \to \mathbb{Z}_{(\infty)}(2)$$

are given by mapping the interval to one of the two diagonals of the lozenge. Thereby, the Waldhausen category structure on free $\mathbb{Z}_{(\infty)}$-modules turns out to be equivalent to the one of finitely generated pointed $\{\pm 1\}$-sets, whose K-theory is well-known. In the course of the proof we also show that other plausible definitions, such as the $S^{-1}S$-construction, the Q-construction, and the $+$-construction yield the same K-groups.

We finish this note by pointing out two K-theoretic differences of the infinite place: we show that $K_0(\mathbb{F}_\infty) = 0$ (Proposition 4.2), as opposed to $K_0(\mathbb{F}_p) = \mathbb{Z}$. Also, the completions at infinity are not well-behaved from a K-theoretic viewpoint. These remarks raise the question whether the "local" ring $\mathbb{Z}_{(\infty)}$ should be considered regular or, more precisely, whether

$$K_0(\mathbb{Z}_{(\infty)}) \to K_0'(\mathbb{Z}_{(\infty)}) := \mathbb{Z}[\text{finitely presented } \mathbb{Z}_{(\infty)}-\mathbf{Mod}]/\text{short exact sequences}$$

is an isomorphism. Unlike in the classical case, there does not seem to be an easy resolution argument in the context of Waldhausen categories. Another natural question is whether there is a Mayer–Vietoris sequence of the form

$$K_i(\widehat{\mathbb{Z}}) \to K_i(\mathbb{Z}) \oplus K_i(\mathbb{Z}_{(\infty)}) \to K_i(\mathbb{Q}) \to K_{i-1}(\widehat{\mathbb{Z}}),$$

where $\widehat{\mathbb{Z}}$ is a generalized scheme obtained by glueing $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathbb{Z}_{(\infty)}$ along $\operatorname{Spec}\mathbb{Q}$. The usual proof of this sequence proceeds by the localization sequence, which is not available in our context.

Throughout the paper, we use the following *notation*: F is a number field with ring of integers $\mathcal{O}_F$. Finite primes of $\mathcal{O}_F$ are denoted by $\mathfrak{p}$. We write Σ_F for the set of real and pairs of complex embeddings of F. The letter σ usually denotes an element of Σ_F. It is referred to as an infinite prime of $\mathcal{O}_F$.

2 Generalized rings

In a few brushstrokes, we recall the definition of generalized rings and their modules and some basic properties. Everything in this section is due to Durov. All references in brackets refer to [1], where a much more detailed discussion is found.

A monad in the category of sets is a functor $R : \mathbf{Sets} \to \mathbf{Sets}$ together with natural transformations $\mu : R \circ R \to R$ and $\epsilon : \mathrm{Id} \to R$ required to satisfy an associativity and unitality axiom akin to the case of monoids. We will write $R(n) := R(\{1, \ldots, n\})$. An R-module is a set X together with a morphism of monads $R \to \mathrm{End}(X)$, where the endomorphism monad $\mathrm{End}(X)$ satisfies $\mathrm{End}(X)(n) = \mathrm{Hom}_{\mathbf{Sets}}(X^n, X)$. In other words, X is endowed with an action

$$R(n) \times X^n \to X$$

satisfying the usual associativity conditions. Thus, $R(n)$ can be thought of as the n-ary operations (acting on any R-module).

Definition 2.1 (*Durov* [*5.1.6*]) A *generalized ring* is a monad R in the category of sets satisfying two additional properties:

- R is *algebraic*, i.e., it commutes with filtered colimits. Since every set is the filtered colimit of its finite subsets, this implies that R is determined by $R(n)$ for $n \geq 0$ [4.1.3].
- R is *commutative*, i.e., for any $t \in R(n)$, $t' \in R(n')$, any R-module X (it suffices to take $X = R(n \times n')$) and $A \in X^{n \times n'}$, we have

$$t(t'(A)) = t'(t(A)),$$

where on the left hand side $t'(A) \in X^n$ is obtained by letting act t' on all rows of A and similarly (with columns) on the right hand side.

For a unital associative ring R (in the sense of usual abstract algebra), let

$$R(S) := \oplus_{s \in S} R$$

be the free R-module of rank $\sharp S$, where S is any set. The addition and multiplication on R turn this into an (algebraic) monad which is commutative iff $R = R(1)$ is [3.4.8]. Indeed, the required map

$$R(1)\times R(1) \to R(1) \tag{1}$$

is just the multiplication in R, while the addition is reformulated as

$$R(2)\times(R(1)\times R(1)) \to R(1), ((x_1, x_2), (y_1, y_2)) \mapsto \sum x_i y_i.$$

Note that (1) is required to exist for any monad, so multiplication is in a sense more fundamental than addition, which requires the particular element $(1, 1) \in R(2)$ [3.4.9].

Reinterpreting a ring as a monad in this way defines a functor from commutative rings to generalized rings, which is easily seen to be fully faithful: given two classical rings R, R', and a map of monads, i.e., a collection of maps $R(n) = R^n \to R'(n) = R'^n$, one checks that the maps for $n \geq 2$ are determined by $R \to R'$. In the same vein, R-modules in the classical sense are equivalent to R-modules (in the generalized sense). Henceforth, we will therefore not distinguish between classical commutative rings and their associated generalized rings.

The initial generalized ring is the monad $\mathbb{F}_0$: **Sets** $\to$ **Sets**, $M \mapsto M$. Its modules are just the same as sets. The monad **Sets** $\ni M \mapsto M \sqcup \{*\}$ is denoted $\mathbb{F}_1$. Neither of these two generalized rings is induced by a classical ring. See Definition 3.2 for our main example of a non-classical ring.

Given a morphism $\phi : R \to S$ of generalized rings, the forgetful functor $\mathbf{Mod}(S) \to \mathbf{Mod}(R)$ between the module categories has a left adjoint $\phi^* : \mathbf{Mod}(R) \to \mathbf{Mod}(S)$ called *base change*. We also denote it by $- \otimes_R S$. Being a left adjoint, this functor preserves colimits [4.6.19]. For example, for a generalized ring R, the unique map $\mathbb{F}_0 \to R$ of generalized rings induces an adjunction

$$\mathbf{Sets} = \mathbf{Mod}(\mathbb{F}_0) \leftrightarrows \mathbf{Mod}(R) : \text{forget}$$

Its left adjoint is explicitly given by $X \mapsto R(X)$, the so-called *free R-module* on some set X. That is,

$$\text{Hom}_{\mathbf{Mod}(R)}(R(X), M) = \text{Hom}_{\mathbf{Sets}}(X, M),$$

as in the classical case.

Coequalizers and arbitrary coproducts exist in $\mathbf{Mod}(R)$, for any generalized ring R [4.6.17]. Therefore, arbitrary colimits exist. Base change functors ϕ^* commute with coequalizers. Moreover, arbitrary limits exist in $\mathbf{Mod}(R)$, and commute with the forgetful functor $\mathbf{Mod}(R) \to$ **Sets** [4.6.1].

An R-module M is called *finitely generated* if there is a surjection $R(n) \twoheadrightarrow M$ for some $0 \leq n < \infty$ [4.6.9]. Unless the contrary is explicitly mentioned, all our modules are supposed to be finitely generated over the ground generalized ring in question. An R-module M is *projective* iff it is a retract of a free module, i.e., if there are maps $M \xrightarrow{i} R(n) \xrightarrow{p} M$ with $pi = \mathrm{id}_M$. As in the classical case this is equivalent to the property that for any surjection of R-modules $N \twoheadrightarrow N'$, $\mathrm{Hom}_{\mathbf{Mod}(R)}(M, N)$ maps onto $\mathrm{Hom}_{\mathbf{Mod}(R)}(M, N')$ [4.6.23]. The categories of (finitely generated) free and projective R-modules are denoted $\mathbf{Free}(R)$ and $\mathbf{Proj}(R)$, respectively.

As usual, an *ideal* I of R is a submodule of $R(1)$. A proper ideal $I \subsetneq R(1)$ is called *prime* if $R(1)\backslash I$ is multiplicatively closed [6.2.2].

3 Archimedean valuation rings

3.1 Definitions

Let K be an integral domain equipped with a norm $|-| : K \to \mathbb{R}^{\geq 0}$. We will write Q for the quotient field of K. We put $E := \{x \in K, |x| = 1\}$. We also write $|x|$ for the L^1-norm on K^n, i.e., $|x| = \sum_i |x_i|$. Throughout, we assume:

Assumption 3.1 (A) $|K^\times| = \{|k|, k \in K^\times\} \subset \mathbb{R}^{\geq 0}$ is dense.
(B) $E \subset K^\times$.

Definition 3.2 The *(generalized) valuation ring* associated to $(K, |-|)$ is the submonad $\mathcal{O}$ of K given by

$$\mathcal{O}(S) := \left\{ x = (x_s) \in \bigoplus_{s \in S} K, |x| := \sum_{s \in S} |x_s| \leq 1 \right\}.$$

This is clearly algebraic. Moreover, the multiplication of the monad, i.e., $\mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ is well-defined by restricting the one of K (and is therefore commutative):

$$\mathcal{O}(\mathcal{O}(n)) = \left\{ (y_x) \in \bigoplus_{x \in \mathcal{O}(n)} K, \sum_x |y_x| \leq 1 \right\} \to \mathcal{O}(n)$$

sends (y_x) to (the finite sum) $\sum_x y_x \cdot x$. A priori, this expression is an element of K^n, only, but is actually contained in $\mathcal{O}(n)$ since

$$\left| \sum_x y_x \cdot x \right| \leq \left(\sum_x |y_x| \right) \cdot \sup |x| \leq 1.$$

In the case of an archimedean valuation, this definition of $\mathcal{O}$ is the one of Durov [1, 5.7.13]. For non-archimedean valuations, Durov's original definition gives back the (generalized ring corresponding to the) ordinary ring $\{x \in K, |x| \leq 1\}$ which is different from Definition 3.2 (see Example 3.4).

By definition, an $\mathcal{O}$-module M is therefore a set such that an expression $\sum_{i=1}^n \lambda_i m_i$ is defined for $n \geq 0$, $m_i \in M$, $\lambda_i \in K$ such that $\sum |\lambda_i| \leq 1$, obeying the usual laws of commutativity, associativity and distributivity. Maps $f : M \to N$ of $\mathcal{O}$-modules are described similarly: they satisfy $f(\sum_i \lambda_i m_i) = \sum_i \lambda_i f(m_i)$. The set $\{0\}$, with its obvious $\mathcal{O}$-module structure is both an initial and terminal $\mathcal{O}$-module. Given a map $f : M' \to M$ of $\mathcal{O}$-modules, the (co)kernel is defined to be the (co)equalizer of the two morphisms f and $M' \to 0 \to M$. As was noted above, the forgetful functor $\mathcal{O} - \mathbf{Mod} \to \mathbf{Sets}$ preserves limits, so the kernel $\ker f$ is just $f^{-1}(0)$. The cokernel is described by the following proposition. Also see Remark 3.11 for an explicit example of a cokernel computation.

Proposition 3.3 *Given a map $f : M' \to M$ of $\mathcal{O}$-modules, the cokernel is given by*

$$coker(f) = M/\sim, \tag{2}$$

where $\sim$ is the equivalence relation generated by $\sum_{i\in I} \lambda_i m_i \sim \sum_{i \in I} \lambda_i \tilde{m}_i$, where I is any finite set, $\lambda = (\lambda_i) \in \mathcal{O}(\sharp I)$ and $m_i, \tilde{m}_i \in M$ are such that either $m_i = \tilde{m}_i$ or both $m_i, \tilde{m}_i \in f(M') \subset M$. This set is endowed with the $\mathcal{O}$-action via the natural projection $\pi : M \to coker(f)$.

Proof This follows from the description of cokernels given in [1, 4.6.13]. It is also easy to check the universal property directly: we clearly have $\pi \circ f = 0$. Given a map $t : M \to T$ of $\mathcal{O}$-modules such that $tf = 0$, we need to see that t factors uniquely through $\operatorname{coker} f$. The unicity of the factorization is clear since $M \to \operatorname{coker} f$ is onto. The existence is equivalent to $t(m_1) = t(m_2)$ whenever $\pi(m_1) = \pi(m_2)$. This is obvious from the definition of the equivalence relation $\sim$ above. □

The base change functor resulting from the monomorphism $\mathcal{O} \subset K$ of generalized rings is denoted

$$(-)_K : \mathbf{Mod}(\mathcal{O}) \to \mathbf{Mod}(K).$$

Actually, using Assumption 3.1, we may pick $t \in K^\times$ such that $|t| < 1$. Then, K is the unary localization $K = \mathcal{O}[1/t]$. This is shown in [1, 6.1.23] for $K = \mathbb{R}$. The proof for a general domain is the same. Therefore K is flat over $\mathcal{O}$, so $(-)_K$ preserves finite limits, in particular kernels [1, 6.1.2, 6.1.8]. Recall from p. 4 that $(-)_K$ also preserves colimits, such as cokernels.

Let $E(n) := \{x \in K(n) = K^n, |x| = 1\}$ be the "boundary" of $\mathcal{O}(n)$. (This is merely a collection of sets, not a monad.) We write $\mathcal{O}$ for $\mathcal{O}(1)$ and E for $E(1)$, if no confusion arises. In particular, $x \in \mathcal{O}$ means $x \in \mathcal{O}(1)$. The i-th standard coordinate vector $e_i = (0, ..., 1, ..., 0)$ is called a *basis vector* of $\mathcal{O}(n)$ $(1 \le i \le n)$.

Example 3.4 Let F be a number field with ring of integers $\mathcal{O}_F$. We fix a complex embedding $\sigma : F \to \mathbb{C}$ and take the norm $|-|$ induced by σ. Let K be either $\mathcal{O}_F[1/N]$ where $N \in \mathbb{Z}$ has at least two distinct prime divisors, or F, or $\widehat{F}^\sigma$, the completion of F with respect to σ. The respective generalized valuation rings will be denoted $\mathcal{O}_{F,1/N,(\sigma)}$, $\mathcal{O}_{F,(\sigma)}$, and $\mathcal{O}_{F,\sigma}$, respectively. For example, $\mathcal{O}_{F,(\sigma)} = \mathcal{O}_{F,(\overline{\sigma})}$. Assumption 3.1(A) is satisfied: for $\mathcal{O}_F[1/N]$, pick two distinct prime divisors $p_1 \neq p_2$ of N. The elements $p_1^{n_1} p_2^{n_2} \in K$ are invertible for any $n_1, n_2 \in \mathbb{Z}$. The subgroup $\{\log(|p_1^{n_1} p_2^{n_2}|), n_i \in \mathbb{Z}\} \subset \mathbb{R}$ is dense: otherwise it was cyclic, in contradiction to the $\mathbb{Q}$-linear independence of $\log p_1$ and $\log p_2$ (Gelfand's theorem).

As for Assumption 3.1(B), let $x \in K$ with $|x| = 1$. If σ is a real embedding, $x = \pm|x| = \pm 1$. If σ is a complex embedding, let $\overline{\sigma}$ be its complex conjugate and $\overline{x} \in K$ be such that $\sigma(\overline{x}) = \overline{\sigma}(x)$. Then $\sigma(x)\sigma(\overline{x}) = \sigma(x)\overline{\sigma(x)} = |\sigma(x)|^2 = 1$ implies $x \in K^\times$.

According to Durov, $\mathcal{O}_{F,(\sigma)}$ is the replacement for infinite places of the local rings $\mathcal{O}_{F(\mathfrak{p})}$ at finite places. However, the analogy is relatively loose, as is shown by the following two remarks: first, for $p < \infty$, let $|x|_p := p^{-v_p(x)}$ for $x \in \mathbb{Q}^\times$. Then the generalized ring $\mathbb{Z}_{|-|_p}$ (in the sense of Definition 3.2) maps injectively to the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime ideal p, but the map is a bijection only in degrees $\leq p$. (Less importantly, Assumption 3.1(A) is not satisfied for $\mathbb{Z}_{|-|_p}$.)

Secondly, recall that the semilocalization $\mathcal{O}_{F(\mathfrak{p}_1,\mathfrak{p}_2)} = \mathcal{O}_{F(\mathfrak{p}_1)} \cap \mathcal{O}_{F(\mathfrak{p}_2)}$ at two finite primes is one-dimensional. In analogy, pick two $\sigma_1, \sigma_2 \in \Sigma_F$ and consider $\mathcal{O} := \mathcal{O}_{(\sigma_1)} \cap \mathcal{O}_{(\sigma_2)} \subset F$, i.e.,

$$\mathcal{O}(n) := \left\{ (x_1, \ldots, x_n) \in F^n, \sum_k |\sigma_i(x_k)| \leq 1 \quad \text{for } i = 1, 2 \right\}.$$

Let $\mathfrak{p}_i = \{x \in \mathcal{O}, |\sigma_i(x)| < 1\}$ and $\mathfrak{p} := \{x \in \mathcal{O}, |\sigma_1(x)\sigma_2(x)| < 1\}$. These are ideals: for example, for $x = (x_j) \in \mathcal{O}(n)$, $s_1, \ldots, s_n \in \mathfrak{p}$, we need to check $\sum s_j x_j \in \mathfrak{p}$: if, say, $|\sigma_1(s_1)| < 1$ then

$$\left| \sigma_1 \left(\sum_j s_j x_j \right) \right| \leq \sum |\sigma_1(s_j)||\sigma_1(x_j)| < \sum |\sigma_1(x_j)| \leq 1.$$

The complement $\mathcal{O} \backslash \mathfrak{p} = \{x, |\sigma_1(x)| = |\sigma_2(x)| = 1\}$ is multiplicatively closed (and contains 1). We get a chain of prime ideals

$$0 \subsetneq \mathfrak{p}_1 \subset \mathfrak{p} \subsetneq \mathcal{O}.$$

The middle inclusion is, in general, strict, namely when $F = \mathbb{Q}[t]/p(t)$ with some irreducible polynomial $p(t)$ having zeros $a_1, a_2 \in \mathbb{C}$ with $|a_1| = 1$, $|a_2| < 1$. That is, $\operatorname{Spec} \mathcal{O}$ is not one-dimensional.

3.2 Projective and free $\mathcal{O}$-modules

In this section we gather a few facts about projective and free $\mathcal{O}$-modules. We begin with a handy criterion for monomorphisms of certain $\mathcal{O}$-modules (Lemma 3.5). Lemma 3.6 concerns a particular unicity property of the basis vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathcal{O}(n)$. This is used to prove Theorem 3.7: every projective $\mathcal{O}$-module is free, provided that the norm is archimedean. This improves a result of Durov which treats only the cases where $\mathcal{O}$ is either the "unclompeted local ring" of a number ring at an infinite place σ, $\mathcal{O}_{F,(\sigma)}$, in the case where σ is a real embedding or the "completed local ring" $\mathcal{O}_{F,\sigma}$ for both real and complex places. Therefore, we only study the K-theory of free $\mathcal{O}$-modules in this paper (but see Remark 3.18). We also use Lemma 3.6 to establish a highly combinatorial flavor of automorphisms of free $\mathcal{O}$-modules (Proposition 3.9), which will later give rise to the computation of higher K-theory of $\mathcal{O}$.

Lemma 3.5 (compare [1, 2.8.3.]) *Let $f : M' \to M$ be a map of $\mathcal{O}$-modules. We suppose both M' and M are submodules of free $\mathcal{O}$-modules. (For example, they might be projective.) Then the following are equivalent:*

a) *$f_Q : M'_Q \to M_Q$ is injective, where Q is the quotient field of K,*
b) *$f_K : M'_K \to M_K$ is injective,*
c) *f is injective (as a map of sets),*
d) *f is a monomorphism of $\mathcal{O}$-modules,*

Proof Consider the diagram

$$\begin{array}{ccccc} M' & \hookrightarrow & M'_K & \hookrightarrow & M'_Q \\ \downarrow f & & \downarrow f_K & & \downarrow f_Q \\ M & \hookrightarrow & M_K & \hookrightarrow & M'_Q. \end{array}$$

Its horizontal maps are injective since both modules are submodules of free modules and, for these, $\mathcal{O}(n) \subset K(n) = K^n \subset Q(n) = Q^n$. This shows (a) $\Rightarrow$ (b) $\Rightarrow$ (c). (c) implies (d) since the forgetful functor $\mathbf{Mod}(\mathcal{O}) \to \mathbf{Sets}$ is faithful. (d) $\Rightarrow$ (b): by Assumption 3.1, we may pick $t \in K^\times$ with $|t| < 1$. Any two element of M'_K are of the form m'_1/t^n, m'_2/t^n, where $m'_1, m'_2 \in M'$ and $n \geq 0$. Suppose that $f_K(m'_1/t^n) = f(m'_1)/t^n$ agrees with $f_K(m'_2/t^n)$. The multiplication with t^{-n} is injective on M'_K, since M' (M'_K) is a submodule of a free $\mathcal{O}$- (K-, respectively) module. Thus $f(m'_1) = f(m'_2)$ so the assumption (d) implies our claim. Finally (b) $\Rightarrow$ (a) follows from the flatness of Q over K. □

The following lemma can be paraphrased by saying that the basis vectors $e_i = (0, \ldots, 1, \ldots 0) \in \mathcal{O}(n)$ cannot be generated as a nontrivial $\mathcal{O}$-linear combination of other elements of $\mathcal{O}(n)$.

Lemma 3.6 *Suppose that K is a field (as opposed to a domain). Suppose further that*

$$e_i = \sum_{j=1}^{m} \lambda_j f_j \tag{3}$$

with $f_j \in \mathcal{O}(n)$ and $(\lambda_j)_j \in \mathcal{O}(m)$, $\lambda_j \neq 0$. Then for each j, $f_j = \mu_j \cdot e_i$ with $\mu_j \in E$.

Proof The proof proceeds by induction on m, the case $m = 1$ being trivial.

Each f_j can be written as $f_j = \sum_{l=1}^{n} \kappa_{jl} e_l$ with $(\kappa_{jl})_l \in \mathcal{O}(n)$. We get

$$1 = |e_i| \overset{(3)}{=} |\sum \lambda_j f_j| \leq \sum |\lambda_j||f_j| \leq \sum |\lambda_j| \leq 1. \tag{4}$$

Therefore equality holds throughout. We have $e_i = \sum_{j,l} \lambda_j \kappa_{jl} e_l$. This K-linear relation between the basis vectors of K^n yields $1 = \sum_j \lambda_j \kappa_{ji}$. Hence

$$1 \leq \sum_j |\lambda_j \kappa_{ji}| \leq \underbrace{\left(\sum |\lambda_j|\right)}_{\overset{(4)}{=}1} \cdot \max_j |\kappa_{ji}|.$$

On the other hand, $|\kappa_{ji}| \leq 1$, so there is some j_0 such that $|\kappa_{j_0 i}| = 1$. Using $\sum_l |\kappa_{j_0 l}| \leq 1$ we see $\kappa_{j_0 l} = 0$ for all $l \neq i$, thus $f_{j_0} = \kappa_{j_0 i} e_i$. Put $\mu_{j_0} := \kappa_{j_0 i} (\in E)$, so

$$(1 - \lambda_{j_0} \mu_{j_0}) e_i = \sum_{j \neq j_0} \lambda_j f_j$$

holds. If $|\lambda_{j_0} \mu_{j_0}| = 1$, we are done since all other λ_j, $j \neq j_0$ must vanish in this case. If $|\lambda_{j_0} \mu_{j_0}| < 1$, then

$$e_i = \sum_{j \neq j_0} \frac{\lambda_j}{1 - \lambda_{j_0} \mu_{j_0}} f_j.$$

This finishes the induction step since the right hand side is actually an $\mathcal{O}$-linear combination of the f_j, for

$$\sum_{j \neq j_0} |\lambda_j| \overset{(4)}{=} 1 - |\lambda_{j_0}| = 1 - |\lambda_{j_0} \mu_{j_0}| \leq |1 - \lambda_{j_0} \mu_{j_0}|.$$

□

Theorem 3.7 *Suppose that the norm $|-|$ giving rise to the generalized valuation ring $\mathcal{O}$ is archimedean. Then every projective $\mathcal{O}$-module M is free.*

Proof Let K' be the completion (with respect to the norm $|-|$) of Q, the quotient field of K. By Ostrowski's theorem, we have either $K' = \mathbb{R}$ or $K' = \mathbb{C}$ (with their usual norms). Let us write $-' := - \otimes_{\mathcal{O}} \mathcal{O}'$, where $\mathcal{O}' := \mathcal{O}_{K'}$ is the generalized valuation ring belonging to K'. We consider the following maps of $\mathcal{O}'$-modules, where O_i are certain free $\mathcal{O}$-modules that are defined in the course of the proof:

$$O'_3 \to O'_2 \to O'_1 \xrightarrow{p'} M' \xrightarrow{\phi, \cong} O'_0.$$

First, M' is a projective $\mathcal{O}'$-module: given a projector $p : O_1 := \mathcal{O}(n_1) \to \mathcal{O}(n_1)$ with $M = \operatorname{im} p$, we get $M' = \operatorname{im} p'$. By the afore-mentioned result of Durov [1, 10.4.2], there is an isomorphism of $\mathcal{O}'$-modules, $\phi : M' \overset{\cong}{\to} O'_0 := \mathcal{O}'(n_0)$. The composition $\phi \circ p'$ is surjective, so for any basis vector $e_i \in O'_0$ $(1 \leq i \leq n_0)$, there is some $\mathcal{O}'$-linear combination $\sum_{j \leq n_1} \lambda_{ij} e_j$ mapping to e_i under $\phi p'$. Thus, $\sum_j \lambda_{ij} \phi p'(e_j) = e_i$. Therefore, by Lemma 3.6, $\phi p'(e_j) \in E' \cdot e_i$ for each j. Here

$E' = \{x \in \mathcal{O}', |x| = 1\}$ (which is $S^1 \subset \mathbb{C}$ or $\{\pm 1\} \subset \mathbb{R}$ depending on K'). We put $O_2 := \sqcup_{j_2 \in J_2} e_{j_2} \mathcal{O} = \mathcal{O}(J_2)$, where the coproduct runs over

$$J_2 := \{1 \leq j_2 \leq n_1, \phi p'(e_{j_2}) \in E' e_i \text{ for some } i \leq n_0\}.$$

The inclusion $J_2 \subset \{1, \dots, n_1\}$ induces a ($\mathcal{O}$-linear!) injection $f_{21} : O_2 \to O_1$. According to the previous remark, $O'_2 \stackrel{\phi p' f'_{21}}{\longrightarrow} O'_1$ is surjective. Consider the map $J_2 \to \{1, \dots, n_0\}$ which maps j_2 to the (unique) i with $e_i \in E' \phi p'(e_{j_2})$. This map is onto. By Assumption 3.1, we may pick some $J_3 \subset J_2$ on which it is a bijection. Let $f_{32} : O_3 := \sqcup_{j_3 \in J_3} e_{j_3} \mathcal{O} = \mathcal{O}(J_3) \to O_2 = \mathcal{O}(J_2)$ be the map induced by $J_3 \subset J_2$. Set $f_{31} = f_{21} \circ f_{32}$. Then the composition $O'_3 \stackrel{f'_{31}}{\rightarrowtail} O'_1 \stackrel{p'}{\to} M' \stackrel{\phi, \cong}{\longrightarrow} O'_0$ is an isomorphism of $\mathcal{O}'$-modules. Note that f_{31} and p are $\mathcal{O}$-linear maps, but ϕ is defined over $\mathcal{O}'$, only. Writing $v := p \circ f_{31}$, we must show the implication

$$v' \text{ isomorphism} \Rightarrow v \text{ isomorphism}.$$

The elements $m_j := p(e_j) \in M$, $j \leq n_1$, generate M. The map $v' \otimes_{\mathcal{O}'} K' = v_Q \otimes_Q K'$ is an isomorphism of K'-vector spaces. The inclusion of the quotient field $Q \to K'$ is fully faithful, so that v_Q is also an isomorphism. Hence there is some $k_j = a_j / b_j \in Q \backslash \{0\}$ such that $k_j m_j \in \operatorname{im} v$. According to Assumption 3.1, we can pick some $N \in K^\times$ such that $|a_j/N|, |b_j/N| \leq 1$ for all j. Then $m_j a_j / N \in \operatorname{im} v$. Similarly, pick some $t \in \mathcal{O}$ with $0 < |t| \leq \min_j |a_j/N|$. Then $tM \subset \operatorname{im} v$.

To show the surjectivity of v, we fix $m \in M$ and pick some $o_3 \in O_3$ with $tm = v(o_3)$. Since $M \subset M'$ and v' is an isomorphism, there is a unique $\tilde{o}'_3 \in O'_3$ with $v'(\tilde{o}'_3) = m$. Hence $v(o_3) = v'(o_3) = v'(t\tilde{o}'_3)$, so that $t\tilde{o}'_3 = o_3$. In other words, $o'_3 = t^{-1} o_3 \in O'_3 \cap (O_3)_K = O_3$. This shows the surjectivity of v. The injectivity of v is clear, since $O_3 \subset O'_3$ and v' is injective. Consequently, v is an isomorphism. □

Definition 3.8 Recall that **Free**($\mathcal{O}$) is the category of (finitely generated) free $\mathcal{O}$-modules. In **Free**($\mathcal{O}$) let *cofibrations* ($\rightarrowtail$) be the monomorphisms whose cokernel (in the category of all $\mathcal{O}$-modules) lies in **Free**($\mathcal{O}$). Morphisms which are obtained as cokernels of cofibrations are called *fibrations* and denoted $\twoheadrightarrow$. Let *weak equivalences* $\stackrel{\sim}{\to}$ be the isomorphisms.

Proposition 3.9 *Let $f : M' \to M$ be a monomorphism of free $\mathcal{O}$-modules with projective cokernel M'' (for example, a cofibration). Then there is a* unique *isomorphism $\phi : M \cong M' \sqcup M''$ such that the following diagram is commutative*

$$\begin{array}{ccccc} M' & \stackrel{f}{\rightarrowtail} & M & \stackrel{\pi}{\twoheadrightarrow} & M''. \\ \| & & \downarrow \phi & & \| \\ M' & \stackrel{incl}{\rightarrowtail} & M' \sqcup M'' & \stackrel{proj}{\twoheadrightarrow} & M''. \end{array} \quad (5)$$

Proof Let $M' = \mathcal{O}(n')$, $M = \mathcal{O}(n)$ and let $f_i := f(e_i) \in M$, $1 \leq i \leq n'$ be the images of the basis vectors.

We claim that f factors through $\sqcup_{i \le n, e_i \in f(M')} e_i \mathcal{O} = \mathcal{O}(\tilde{n}') \subset M = \mathcal{O}(n)$, where $\tilde{n}' := \sharp\{i \le n, e_i \in f(M')\}$. To show this, write $f(M') \ni m' = \sum_{i \in I} \lambda_i e_i$, where all $\lambda_i \neq 0$ and the e_i are the basis vectors of M. Put

$$m' = \underbrace{\sum_{e_i \notin f(M')} \lambda_i e_i}_{=:m'_1} + \underbrace{\sum_{e_i \in f(M')} \lambda_i e_i}_{=:m'_2} .$$

By Assumption 3.1, we can pick some $t \in K^\times$ such that $|t| \le 1/2$. Then $tm'_1 = tm' - tm'_2 \in f(M')$. Let i be such that $e_i \notin f(M')$. We need to see $\lambda_i = 0$.

We write $(-)_Q$ for the functor $- \otimes_{\mathcal{O}} \mathcal{O}_Q$, where $\mathcal{O}_Q$ is the generalized valuation ring associated to the unique extension of the norm $|-|$ in K to the quotient field Q of K. The functor $(-)_Q$ preserves colimits, in particular $\operatorname{coker}(f_Q) = (\operatorname{coker} f)_Q$. In addition, f_Q is a monomorphism by Lemma 3.5. The assumption $e_i \notin f(M')$ implies $e_i \notin f_Q(M'_Q)$: suppose that $e_i = \sum_{i' \le n'} \kappa_{i'} f_{i'}$ where $(\kappa_{i'}) \in \mathcal{O}_Q(n')$ and $f_{i'} := f(e_{i'})$ are the images of the basis vectors of M'. By Lemma 3.6, we have $f_{i'} = \epsilon_{i'} e_i$ for all i', with some $\epsilon_{i'} \in \mathcal{O}_Q$, $|\epsilon_{i'}| = 1$. But $f_{i'}$ also lies in M (as opposed to M_Q). Thus, $\epsilon_{i'}$ must lie in $\mathcal{O}$, that is, $e_i \in f(M')$. Therefore, to prove the claim we may assume K is a field.

Now, by Lemma 3.6, e_i is not a non-trivial $\mathcal{O}$-linear combination of other elements of M. As $e_i \notin f(M')$, Proposition 3.3 implies

$$\pi^{-1}(\pi(e_i)) = \{e_i\}. \tag{6}$$

Fix a section $\sigma : M'' \to M$ of π, which exists by the assumption that M'' be projective. We obtain $\sigma(\pi(e_i)) = e_i$. Hence,

$$0 = \sigma(0_{M''}) = \sigma(\pi(tm'_1)) = \sum_{e_i \notin f(M')} t\lambda_i \sigma(\pi(e_i)) = \sum_{e_i \notin f(M')} t\lambda_i e_i ,$$

so that $\lambda_i = 0$. The claim is shown.

By the claim, f induces a bijection $\tilde{f} : M' = \mathcal{O}(n') \to \mathcal{O}(\tilde{n}')$, which gives rise to a bijection $K^{n'} \to K^{\tilde{n}'}$. This shows $\tilde{n}' = n'$. We conclude that the basis vectors $e_i \in M'$ get mapped under f to $\epsilon_i e_{J(i)}$ where $\epsilon_i \in E$ and $J : \{1, \dots, n'\} \to \{1, \dots, n\}$ is an injective set map. In fact, suppose $\tilde{f}^{-1}(e_i) = \sum_{j \in J} \lambda_{ij} e_j$ with $(\lambda_{ij}) \in \mathcal{O}(J)$ with all $\lambda_{ij} \neq 0$. Equivalently, $\sum \lambda_{ij} \tilde{f}(e_j) = e_i$. Therefore, by Lemma 3.6 (applied with Q instead of K), $\tilde{f}_Q(e_j) \in E_Q \cdot e_i$ for all j, where $E_Q = \{q \in Q, |q| = 1\}$. Since $\tilde{f}$ and therefore, by Lemma 3.5, $\tilde{f}_Q$ is injective, this implies that only one summand appears in this sum, i.e., $\tilde{f}(e_j) = \lambda_{ij}^{-1} e_i$ for some $j \in J$. A priori, λ_{ij}^{-1} only lies in Q, but $\tilde{f}(e_j) \in \mathcal{O}(n')$ shows that $\epsilon_i := \lambda_{ij}^{-1} \in \mathcal{O}$, hence in E.

By Assumption 3.1, $\epsilon_i \in E$ is a unit in K. We can therefore define $\phi' : \mathcal{O}(n') \to M'$ by mapping the basis vectors e_i of $\mathcal{O}(n')$ (which correspond, in the above notation, to the basis vectors $e_{J(i)}$ of M) to $\epsilon_i^{-1} e_i$. Also, let $\phi'' : \mathcal{O}(n - n') \subset M \to M''$ be the map which sends the remaining basis vectors $e_{j'}$ for $j' \notin \mathrm{im} J$ to $\pi(e_{j'})$. Put

$$\phi := \phi' \sqcup \phi'' : M = \mathcal{O}(n) = \mathcal{O}(n') \sqcup \mathcal{O}(n - n') \to M' \sqcup M''.$$

Both ϕ' and ϕ'' are onto, hence so is ϕ. This follows from the construction of coproducts of modules over generalized rings [1, 4.6.15]. (Also see [1, 10.4.7] for an explicit description of the coproduct for modules over archimedean valuation rings.) Alternatively, the surjective maps ϕ' and ϕ'' are epimorphisms of $\mathcal{O}$-modules. Hence their coproduct ϕ is an epimorphism. As $M' \sqcup M''$ is projective, ϕ has a section, so it is also surjective. The map ϕ is injective, as can be seen by checking the definition or using Lemma 3.5(b) $\Rightarrow$ (c). Hence ϕ is an isomorphism.

We finally show the unicity of ϕ or, in other words, that there are no non-trivial automorphism of cofiber sequences

$$0 \to M' \rightarrowtail M \twoheadrightarrow M'' \to 0.$$

Suppose $\tilde{\phi}$ is another isomorphism fitting into (5). We replace ϕ by $\tilde{\phi}\phi^{-1}$ and $\tilde{\phi}$ by id_M and assume f is the standard inclusion $M' \to M = M' \sqcup M''$ and π is the standard projection onto M''. Applying the base change functor $(-)_Q$ (see above), we may assume that K is a field. Then M''_K is a free K-module, so the endomorphism $\phi_K : M_K \to M_K$ is given by a matrix

$$B = \begin{pmatrix} \mathrm{Id}_{M'} & A \\ 0 & \mathrm{Id}_{M''} \end{pmatrix},$$

where A is the matrix corresponding to the map $M''_K \to M'_K$ (of free K-modules). On the other hand, ϕ is a map of free $\mathcal{O}$-modules, so every column in B is in $\mathcal{O}(n)$. This forces $A = 0$, so that $\phi = \mathrm{id}_M$. □

Theorem 3.10 *The category* $(\mathbf{Free}(\mathcal{O}), \rightarrowtail, \xrightarrow{\sim})$ *defined in 3.8 is a Waldhausen category.*

Proof The only non-trivial thing to show is the stability of cofibrations under cobase-change. By Proposition 3.9, a cofibration sequence $M' \overset{\iota}{\rightarrowtail} M \overset{\pi}{\twoheadrightarrow} M''$ in $\mathbf{Free}(\mathcal{O})$ is isomorphic to $M' \rightarrowtail M' \sqcup M'' \twoheadrightarrow M''$. Hence, given any map $f : M' \to \tilde{M}'$, the pushout of ι along f, $\tilde{M}' \to \tilde{M}' \sqcup_{M'} M$ is isomorphic to $\tilde{M}' \to \tilde{M}' \sqcup M''$ which is a monomorphism with cokernel M''. □

Remark 3.11 Mahanta uses split monomorphisms as cofibrations in the category of finitely generated modules over a fixed $\mathbb{F}_1$-algebra (i.e., pointed monoid) to define G- (a.k.a. K'-)theory of such algebras [3]. In $\mathbf{Free}(\mathcal{O})$, we have seen that all cofibrations are split, but not conversely: the cokernel of the split monomorphism $\varphi : \mathbb{Z}_\infty(1) \to \mathbb{Z}_\infty(2)$, $e_1 \mapsto \frac{e_1}{2} + \frac{e_2}{2}$ is not free. This follows either from Proposition 3.9 or by an explicit computation, using Proposition 3.3. Indeed, two elements $x_i e_1 + y_i e_2 \in \mathbb{Z}_\infty(2)$ $(i = 1, 2)$ are identified in $\mathrm{coker}\varphi$ iff $|y_1 - x_1| = |y_2 - x_2| < 1$. On $\mathrm{coker}\varphi$, multiplication with $1/2$ is therefore not injective. Thus $\mathrm{coker}\varphi$ is not a submodule of a free $\mathbb{Z}_\infty$-module, in particular it is not projective.

3.3 K-theory

In this subsection, we compute the K-theory of the generalized valuation ring $\mathcal{O}$ (Definition 3.2) or, more precisely, of the category of free $\mathcal{O}$-modules. By Theorem 3.7, every projective $\mathcal{O}$-module is free, provided that the norm is archimedean.

We define the K-theory using Waldhausen's $S_\bullet$-construction, which has the advantage of being immediately applicable (Theorem 3.10). Other constructions, such as Quillen's Q-construction can also be applied (slightly modified, since $\mathcal{O}$-modules do not form an exact category). The resulting K-groups do not depend on the choice of the construction.

Recall the definition of K-theory of a Waldhausen category $\mathcal{C}$ (see e.g. [7, Section IV.8] for more details). We always assume that the weak equivalences of $\mathcal{C}$ are its isomorphisms. The category $S_n\mathcal{C}$ consists of diagrams

$$\begin{array}{ccccccccc}
0 = A_{00} & \rightarrowtail & A_{01} & \rightarrowtail & A_{02} & \rightarrowtail & \cdots & \rightarrowtail & A_{0n} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 0 = A_{11} & \rightarrowtail & A_{12} & \rightarrowtail & \cdots & \rightarrowtail & A_{1n} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & 0 = A_{22} & \rightarrowtail & \cdots & \rightarrowtail & A_{2n} \\
 & & & & & & & & \downarrow \\
 & & & & & & \ddots & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & & & A_{n-1,n}
\end{array} \tag{7}$$

such that $A_{i,j} \rightarrowtail A_{i,k} \twoheadrightarrow A_{j,k}$ is a cofibration sequence. Varying n yields a simplicial category $S_\bullet\mathcal{C}$. The subcategory of isomorphisms is denoted $wS_\bullet\mathcal{C}$. Applying the classifying space construction of a category yields a pointed bisimplicial set $S(\mathcal{C})_{n,m} := B_m wS_n\mathcal{C}$. For example, $S(\mathcal{C})_{n,0} = \mathrm{Obj}(S_n\mathcal{C})$. The K-theory of $\mathcal{C}$ is defined as

$$K_i(\mathcal{C}) := \pi_{i+1} d(B_* wS_\bullet\mathcal{C}),$$

where $d(-)$ is the diagonal of a bisimplical set.

By Theorem 3.10, we are ready to define the *algebraic K-theory* of $\mathcal{O}$. More precisely, we consider the Waldhausen category of (finitely generated) free $\mathcal{O}$-modules, which is the same as projective $\mathcal{O}$-modules in all cases of interest by Theorem 3.7.

Definition 3.12

$$K_i(\mathcal{O}) := K_i(\mathbf{Free}(\mathcal{O})) = \pi_{i+1}(dBwS_\bullet\mathbf{Free}(\mathcal{O})), \quad i \geq 0.$$

Lemma 3.13 *Given two normed domains and a ring homomorphism $f : K \to K'$ between them satisfying $|f(x)| = |x|$ (so that f restricts to a map $f : \mathcal{O} \to \mathcal{O}'$), the functor $f^* : \mathbf{Free}(\mathcal{O}) \to \mathbf{Free}(\mathcal{O}')$, $M \mapsto M \otimes_{\mathcal{O}} \mathcal{O}'$ is (Waldhausen-)exact and therefore induces a functorial map*

$$f^* : K_i(\mathcal{O}) \to K_i(\mathcal{O}').$$

Proof As pointed out at p. 4, $f^* : \mathbf{Mod}(\mathcal{O}) \to \mathbf{Mod}(\mathcal{O}')$ preserves cokernels. Secondly, tensoring with $\mathcal{O}'$ preserves cofibrations since a map $M \to M'$ of free (or projective) $\mathcal{O}$-modules is a monomorphism iff $M_Q \to M'_Q$ is one (where Q is the quotient field of K, Lemma 3.5) and the statement is true for Q-modules: the map $Q \to Q'$ is injective since $|f(1)| = |1| = 1$ and therefore flat. □

The group $K_0(\mathcal{O})$ is the free abelian group generated by the isomorphisms classes of free $\mathcal{O}$-modules modulo the relations

$$[\mathcal{O}(n') \sqcup \mathcal{O}(n'')] = [\mathcal{O}(n')] + [\mathcal{O}(n'')].$$

Indeed, any cofiber sequence satisfies additivity of the ranks of the involved free modules, as one sees by tensoring the sequence with the quotient field Q of K. Therefore, $K_0(\mathcal{O}) = \mathbb{Z}$.

We now turn to higher K-theory of $\mathcal{O}$. Recall that $E := \{x \in \mathcal{O}, |x| = 1\}$ is the subgroup of norm one elements. Let us write $\mathrm{GL}_n(\mathcal{O}) := \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}(n))$. According to Proposition 3.9,

$$\mathrm{GL}_n(\mathcal{O}) = E \wr S_n = E^n \rtimes S_n, \tag{8}$$

where the symmetric group S_n acts on E^n by permutations. For $E = \mu_2 = \{\pm 1\}$, this group is known as the *hyperoctahedral group*. As usual, we write

$$\mathrm{GL}(\mathcal{O}) := \varinjlim_n \mathrm{GL}_n(\mathcal{O})$$

for the infinite linear group, where the transition maps are induced by $\mathrm{GL}_n(\mathcal{O}(n) \ni f \mapsto f \sqcup \mathrm{id}_{\mathcal{O}}$. For any group G, let $G_{\mathrm{ab}} = G/[G, G]$ be its abelianization. We write $\pi_i^{\mathrm{s}}(-)$ for the stable homotopy groups of a space and abbreviate $\pi_i^{\mathrm{s}} := \pi_i^{\mathrm{s}}(S^0)$.

Theorem 3.14 *Let $\mathcal{O}$ be a generalized valuation ring as defined in 3.2. Then for $i \geq 0$, there is an isomorphism*

$$K_i(\mathcal{O}) \cong \pi_i^{\mathrm{s}}(BE_+, *),$$

where the right hand side denotes the i-th stable homotopy group of the classifying space of E (viewed as a discrete group), with a disjoint base point $*$. For a map f as in Lemma 3.13, this isomorphism identifies f^* in K-theory with the map on stable homotopy groups induced by $E(\mathcal{O}) \to E(\mathcal{O}')$.

For $i = 1, 2$ we get

$$K_1(\mathcal{O}) = \mathrm{GL}(\mathcal{O})_{\mathrm{ab}} = E \times \mathbb{Z}/2$$
$$K_2(\mathcal{O}) = \varinjlim_n \mathrm{H}_2([\mathrm{GL}_n(\mathcal{O}), \mathrm{GL}_n(\mathcal{O})], \mathbb{Z}) \tag{9}$$

where the right hand side in (9) is group homology with $\mathbb{Z}$-coefficients.

Before proving the theorem, we first discuss our main example, when $\mathcal{O}$ comes from an infinite place of a number field, as in Example 3.4. Then, we prove a preliminary lemma.

Example 3.15 Let us consider a number field F with the norm induced by some complex embedding $\sigma \in \Sigma_F$ (see p. 3 for notation). The torsion subgroup E_{tor} of $E := \{x \in F^\times, |x| = 1\}$ agrees with the finite group μ_F of roots of unity. The exact localization sequence involving all finite primes of $\mathcal{O}_F$,

$$1 \to \mathcal{O}_F^\times \to F^\times \to L := \ker(\oplus_{\mathfrak{p}<\infty}\mathbb{Z} \to \mathrm{cl}(F)) \to 0,$$

shows $F^\times/\mu_F \cong \mathcal{O}_F^\times/\mu_F \oplus L$. Hence it is free abelian by Dirichlet's unit theorem. Thus

$$E \subset \mu_F \oplus \mathbb{Z}^{r_1+r_2-1} \oplus L,$$

where r_1 and r_2 are the numbers of real and pairs of complex embeddings. Therefore, $E = \mu_F \oplus \mathbb{Z}^S$, where $S := \mathrm{rk}E$ is at most countably infinite. Of course, $E = \{\pm 1\}$ whenever σ is a real embedding, but also, for example, for any complex embedding of $F = \mathbb{Q}[\sqrt[3]{2}]$. For $F = \mathbb{Q}[\sqrt{-1}]$, E is the (countably) infinitely generated group of pythagorean triples [2] (see also [8] for a description of the group structure of pythagorean triples in more general number fields).

The group μ_F is cyclic of order w, so the long exact sequence of group homology,

$$\mathrm{H}_i(\mu_F, \mathbb{Z}) \xrightarrow{\cdot n} \mathrm{H}_i(\mu_F, \mathbb{Z}) \to \mathrm{H}_i(\mu_F, \mathbb{Z}/n) \to \mathrm{H}_{i-1}(\mu_F, \mathbb{Z}),$$

together with the Atiyah–Hirzebruch spectral sequence

$$\mathrm{H}_p(\mu_F, \pi_q^{\mathrm{s}}) = \mathrm{H}_p(B\mu_F, \pi_q^{\mathrm{s}}) \Rightarrow \pi_{p+q}^{\mathrm{s}}(B\mu_F) = \pi_{p+q}^{\mathrm{s}}((B\mu_F)_+, *)$$

yield at least for small p and q explicit bounds on $\pi_{p+q}^{\mathrm{s}}((B\mu_F)_+, *)$: the E^2-page reads

$q \uparrow$				
2	$\pi_2^{\mathrm{s}} = \mathbb{Z}/2$	$\mathbb{Z}/w'$	$\mathbb{Z}/w'$	
1	$\pi_1^{\mathrm{s}} = \mathbb{Z}/2$	$\mu_F/2 = \mathbb{Z}/w'$	$\mathbb{Z}/w'$	
0	$\mathbb{Z}$	$\mu_F = \mathbb{Z}/w$	0	
	0	1	2	$p \rightarrow$

where $w' = (2, w)$. In general, $\pi^{s}_{p+q}((B\mu_F)_+, *)$ is finite for $p + q > 0$. For $i > 0$,

$$\begin{aligned} K_i(\mathcal{O}_{F\sigma}) = K_i(\mathcal{O}_{F(\sigma)}) &= \pi^s_i(B(\mu_F \oplus \mathbb{Z}^{\oplus S})_+, *) \\ &= \pi^s_i\left((B\mu_F)_+ \vee \bigvee_S S^1, *\right) \\ &= \pi^s_i(B\mu_F) \oplus \bigoplus_S \pi^s_{i-1}. \end{aligned}$$

In particular

$$\begin{aligned} K_1(\mathcal{O}_{F(\sigma)}) &= \mathbb{Z}/2 \oplus \mu_F \oplus \mathbb{Z}^{\oplus S}, \\ K_2(\mathcal{O}_{F(\sigma)}) &= G \oplus (\mathbb{Z}/2)^{\oplus S}, \end{aligned}$$

where G is a finite (abelian) group which is filtered by a filtration whose graded pieces are subquotients of $\mathbb{Z}/2$ and $\mathbb{Z}/w'$. (Determining G would require studying the differentials of the spectral sequence).

Lemma 3.16 *The map*

$$\mathrm{GL}(\mathcal{O})_{\mathrm{ab}} \to E \times \mathbb{Z}/2, (\epsilon, \sigma) \mapsto \left(\prod_{i=1}^{\infty} \epsilon_i, parity(\sigma)\right)$$

is an isomorphism. Here the representation of elements of $\mathrm{GL}(\mathcal{O})$ *is as in (8). The group* $[\mathrm{GL}(\mathcal{O}), \mathrm{GL}(\mathcal{O})]$ *is perfect.*

Proof For $i \geq 1$ and $\epsilon \in E$, let $\epsilon_i = (1, \ldots, 1, \epsilon, 1, \ldots) \in E \times E \times \ldots$ be the vector with ϵ at the i-th spot. Let $\sigma_i = (i, i+1) \in S_n$ be the permutation swapping the i-th and $i+1$-st letter. The ϵ_i and σ_i, for $i \geq 1$ and $\epsilon \in E$, generate $G := \mathrm{GL}(\mathcal{O})$ as we have seen in the proof of Proposition 3.9. In G, we have relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, which implies $\sigma_i = \sigma_{i+1}$ in G_{ab}. Moreover, in G we have the relation $\epsilon_i \sigma_i = \sigma_{i+1} \epsilon_{i+1}$, so that we get $\epsilon_i = \epsilon_{i+1}$ in G_{ab}. This shows the first claim.

The perfectness of $[\mathrm{GL}(\mathcal{O}), \mathrm{GL}(\mathcal{O})]$ is a special case of [6, Prop. 3], for example. Alternatively, the above implies that $H := [\mathrm{Aut}(\mathcal{O}(n)), \mathrm{Aut}(\mathcal{O}(n))]$ is given by $H = L \rtimes A_n$, where the alternating group A_n acts on $L := \ker(\prod_{i=1}^n E \to E, (\epsilon^1, \ldots, \epsilon^n) \mapsto \prod \epsilon^i) (\cong E^{n-1})$ by restricting the S_n-action on E^n. Now, the perfectness of A_n for $n \geq 5$ and a simple explicit computation shows $H_{\mathrm{ab}} = 1$ for $n \geq 5$. □

We now prove Theorem 3.14. This theorem is actually an immediate consequence of Proposition 3.9, together with well-known facts about K-theory of G-sets, where G is some group [7, Ex. IV.8.9]. For example, the K-theory of the Waldhausen category of finite pointed sets (which would correspond to the impossible case $E = 1$) is

$$K_i(\mathbb{F}_1) := K_i((\text{finite pointed sets, injections, bijections})) = \pi^s_i,$$

the stable homotopy groups of spheres. More generally, for some (discrete) group G, the K-theory of the category $\mathbf{Free}(G)$ of finitely generated (i.e., only finitely many orbits) pointed G-sets on which the G-action is fixed-point free, together with bijections as weak equivalences and injections as cofibrations, is known to be the stable homotopy group of $(BG)_+$. By Proposition 3.9, the canonical functor

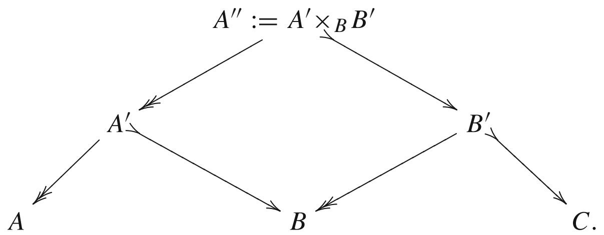
$$\mathbf{Free}(E) \to \mathbf{Free}(\mathcal{O}), (E^X) \sqcup \{*\} \mapsto \mathcal{O}(X)$$

induces an equivalence of the categories of cofibrations and therefore an isomorphism of K-theory. For the convenience of the reader, we recall the necessary arguments, which also includes showing that other definitions of higher K-theory (of free $\mathcal{O}$-modules) yield the same K-groups.

Proof Let $Q\mathbf{Free}(\mathcal{O})$ be Quillen's Q-construction, i.e., the category whose objects are the ones of $\mathbf{Free}(\mathcal{O})$ and

$$\mathrm{Hom}_{Q\mathbf{Free}(\mathcal{O})}(A, B) := \{A \twoheadleftarrow A' \rightarrowtail B\}/\sim,$$

where two such roofs are identified if there is an isomorphism between them which is the identity on A and B. It forms a category whose composition is given by the composite roof defined by the cartesian diagram



Here, we use that A'' exists (in $\mathbf{Free}(\mathcal{O})$) since it is the kernel of the composite $B' \twoheadrightarrow B \twoheadrightarrow B/A'$, which is split by Proposition 3.9. The subcategory $S := \mathrm{Iso}(\mathbf{Free}(\mathcal{O}))$ of $\mathbf{Free}(\mathcal{O})$ consisting of isomorphisms only is a monoidal category under the coproduct. Hence $S^{-1}S$ is defined. We claim

$$\Omega B Q\mathbf{Free}(\mathcal{O}) = B(S^{-1}S).$$

Indeed, the proof of [7, Theorem IV.7.1] carries over: the extension category $\mathcal{E}\mathbf{Free}(\mathcal{O})$ is defined as in *loc. cit.* and comes with a functor $t : \mathcal{E}\mathbf{Free}(\mathcal{O}) \to Q\mathbf{Free}(\mathcal{O})$, $(A \rightarrowtail B \twoheadrightarrow C) \mapsto C$. The fiber $\mathcal{E}_C := t^{-1}C$ ($C \in \mathbf{Free}(\mathcal{O})$) consists of sequences $A \rightarrowtail B \twoheadrightarrow C$. The functor

$$\phi : S \to \mathcal{E}_C, \quad A \mapsto A \rightarrowtail A \sqcup C \twoheadrightarrow C$$

induces a homotopy equivalence $B(S^{-1}S) \to B(S^{-1}\mathcal{E}_C)$ in the classical case of an exact category (instead of $\mathbf{Free}(\mathcal{O})$). In our situation, ϕ is an equivalence of categories since any extension in $\mathbf{Free}(\mathcal{O})$ splits *uniquely* (Proposition 3.9). Thus [7, Theorem IV.4.10] gives

$$BQ\mathbf{Free}(\mathcal{O}) = K_0(S) \times B\mathrm{GL}(\mathcal{O})^+,$$

where the right hand side is the $+$-construction with respect to the perfect normal subgroup $[\mathrm{GL}(\mathcal{O}), \mathrm{GL}(\mathcal{O})]$ (Lemma 3.16). In the same vein, Waldhausen's comparison of the Q-construction and his $S_\bullet$-construction carries over: $d(BwS_\bullet\mathbf{Free}(\mathcal{O}))$ is weakly equivalent to $BQ\mathbf{Free}(\mathcal{O})$.

Finally, by the Barratt–Priddy theorem (see e.g. [5, Th. 3.6])

$$\pi_i(B\mathrm{GL}(\mathcal{O})^+) \cong \pi_i^s(BE_+, *).$$

The identification of the low-degree K-groups is the standard calculation of the $S^{-1}S$-construction [7, IV.4.8.1, IV.4.10]. □

Remark 3.17 The calculation of $K_1(\mathcal{O})$ could also be done using the description of K_1 of a Waldhausen category due to Muro and Tonks [4].

Remark 3.18 Recall that for an (ordinary) ring R the following two properties of an R-module M are equivalent: (i) it is projective, (ii) there is another projective module M' such that $M \sqcup M'$ is free. I have not been able to show the corresponding statement for projective $\mathcal{O}$-modules. For example, for a projector $p : \mathcal{O}(n) \to \mathcal{O}(n)$ with $M = \mathrm{im} p$, it is *not* true that the canonical map

$$\phi : M \sqcup \ker p \to \mathcal{O}(n)$$

is an isomorphism of $\mathcal{O}$-modules: for $n = 2$ and the projector p given by the matrix

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

$\ker p$ is the free $\mathcal{O}$-module of rank 1, generated by $(e_1 - e_2)/2 \in \mathcal{O}(2)$. In this case, ϕ induces an isomorphism of $M \sqcup \ker p$ with the free $\mathcal{O}$-module of rank 2 generated by $(e_1 \pm e_2)/2$, but not with $\mathcal{O}(2) = (e_1, e_2)$. The analogous statement of Proposition 3.9 for cofibrations of projective $\mathcal{O}$-modules, as well as the computation of $K_i(\mathbf{Proj}(\mathcal{O}))$ for $i > 0$ (using Waldhausen's cofinality theorem) would carry over verbatim if the above statement about projective $\mathcal{O}$-modules holds. However, the distinction between projective and free modules is only relevant for non-archimedean valuations, by Theorem 3.7.

4 The residue field at infinity

We finish this work by noting two differences (as far as K-theory is concerned) to the case of classical rings, namely the K-theory of the residue "field" at infinity, and the behavior with respect to completion. For simplicity, we restrict our attention to the case $F = \mathbb{Q}$.

Let $p < \infty$ be a (rational) prime with residue field $\mathbb{F}_p$. There is a long exact sequence

$$K_n(\mathbb{F}_p) \to K_n(\mathbb{Z}_{(p)}) \to K_n(\mathbb{Q}) \xrightarrow{\delta} K_{n-1}(\mathbb{F}_p)$$

which stems from the fact that $\mathbb{Z}_{(p)}$ (the localization of $\mathbb{Z}$ at the prime ideal (p)) is a Noetherian regular local ring of dimension one. Moreover, for $n = 1$ the map δ is the p-adic valuation $v_p : \mathbb{Q}^\times \to \mathbb{Z}$. The situation is less formidable at the infinite places, as we will now see. The (generalized) valuation ring $\mathbb{Z}_{(\infty)}$ (Definition 3.2) is *not* Noetherian: ascending chains of ideals need not terminate. Indeed, consider a finitely generated ideal $I = (m_1, \dots, m_n) \subset \mathbb{Z}_{(\infty)}$. Then $|I| = \{|m|, m \in I\} = [0, \max_i |m_i|] \cap |\mathbb{Z}_{(\infty)}|$. In particular, an ideal of the form $\{x \in \mathbb{Z}_{(\infty)}, |x| < \lambda\}, \lambda \leq 1$ is not finitely generated, since $|\mathbb{Z}_{(\infty)}|$ is dense in $[0, 1]$. This should be compared with the well-known fact that the valuation ring of a non-archimedean field is noetherian iff the field is trivially or discretely valued.

Definition 4.1 [1, 4.8.13] Put $\mathbb{F}_\infty := \mathbb{Z}_{(\infty)}/\widetilde{\mathbb{Z}_{(\infty)}}$, where $\widetilde{\mathbb{Z}_{(\infty)}}$ is the submonad given by

$$\widetilde{\mathbb{Z}_{(\infty)}}(n) = \{x \in \mathbb{Q}^n, |x| < 1\}.$$

We refer to *loc. cit.* for the general definition of strict quotients of generalized rings by appropriate relations. For us, it is enough to note that every element of $\mathbb{Z}_{(\infty)}(n)$ is uniquely represented by $z = \sum_{i \in I} \lambda_i \epsilon_i e_i$, where $I \subset \{1, \dots, n\}$, $0 < \lambda_i \leq 1$, $\sum \lambda_i \leq 1$, $\epsilon_i \in E_{\mathbb{Z}_{(\infty)}} = \{\pm 1\}$, and e_i is the standard basis vector. Two elements $z, z' \in \mathbb{Z}_{(\infty)}(n)$ get identified in $\mathbb{F}_\infty(n)$ (Notation: $z \equiv z'$) iff

$$|z| < 1 \quad \text{and} \quad |z'| < 1 \tag{10}$$

or

$$|z| = |z'| = 1,\ I_z = I_{z'}, \quad \text{and} \quad \epsilon_{i,z} = \epsilon_{i,z'} \quad \text{for all } i \in I_z. \tag{11}$$

That is, as a set $\mathbb{F}_\infty(n)$ consists of the faces of the n-dimensional octahedron. Again, 0 is the initial and terminal $\mathbb{F}_\infty$-module, so we can speak about (co)kernels.

As usual, we put

$$K_0(\mathbb{F}_\infty) := \left(\bigoplus_{M \in \mathbf{Free}(\mathbb{F}_\infty)/Iso} \mathbb{Z} \right) / [M] = [M'] + [M''],$$

with a relation for each monomorphism $M' \to M$ in $\mathbf{Free}(\mathbb{F}_\infty)$ such that its cokernel M'' (computed in $\mathbf{Mod}(\mathbb{F}_\infty)$) lies in $\mathbf{Free}(\mathbb{F}_\infty)$. Similarly, we define $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty)$ using projective $\mathbb{F}_\infty$-modules. Using the above, one sees that $\mathbb{F}_\infty$ is not finitely presented as a $\mathbb{Z}_{(\infty)}$-module. Thus, one should not expect a natural map $i_* : K_0(\mathbb{F}_\infty) \to K_0(\mathbb{Z}_{(\infty)})$. Actually, K-theory of $\mathbb{F}_\infty$-modules behaves badly in the sense of the following proposition:

Proposition 4.2 *$K_0^{\mathbf{Proj}}(\mathbb{F}_\infty) = 0$, $K_0(\mathbb{F}_\infty) = \mathbb{Z}$. In particular, there is no exact* localization sequence *(regardless of the maps involved)*

$$\begin{aligned} K_1(\mathbb{Z}_{(\infty)}) = \mathbb{Z}/2 \times \{\pm 1\} \to K_1(\mathbb{Q}) = \mathbb{Q}^\times \to K_0(\mathbb{F}_\infty) \to K_0(\mathbb{Z}_{(\infty)}) \\ = \mathbb{Z} \to K_0(\mathbb{Q}) = \mathbb{Z}, \end{aligned}$$

or similarly with $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty)$ instead.

Proof We first show that any projective $\mathbb{F}_\infty$-module M which is generated by n elements contains $\mathbb{F}_\infty$ as a submodule, such that the cokernel is a projective $\mathbb{F}_\infty$-module generated by $n-1$ elements. This implies that $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty)$ is generated by $[\mathbb{F}_\infty]$ (which is obvious for $K_0(\mathbb{F}_\infty)$).

The projective module M is specified by a projector $\pi : \mathbb{F}_\infty(n) \to \mathbb{F}_\infty(n)$ with $M = \pi(\mathbb{F}_\infty(n))$. Let $a_i := \pi(e_i) \in \mathbb{F}_\infty(n)$. We pick $a_{ij} \in [-1, 1] \subset \mathbb{R}$ such that $a_i \equiv \sum_{j \in J_i} a_{ij} e_j$ with $a_{ij} \neq 0$ for all $j \in J_i$. Set $A := (a_{ij}) \in \mathbb{R}^{n \times n}$. We may assume that the number n of generators of M is minimal, i.e., there is no surjection $p' : \mathbb{F}_\infty(n') \to M$ with $n' < n$. Indeed, if there is such a surjection, it has a section σ' since M is projective, and $\pi' := \sigma' p'$ would again be a projector.

The minimality of n implies that $a_i \not\equiv a_j$ for all $i \neq j$. Otherwise, the restriction of π to $\mathbb{F}_\infty(n \backslash \{i\}) \subset \mathbb{F}_\infty(n)$ would be surjective. Similarly, the minimality implies $a_i \not\equiv 0 \in \mathbb{F}_\infty(n)$ for all i. Also, put $B = (b_{ij}) := A^2 \in \mathbb{R}^{n \times n}$. Using $(b_{ij})_j \equiv \pi(a_i) \equiv a_i \not\equiv 0 \in \mathbb{F}_\infty(n)$, we obtain $\sum_j |b_{ij}| = 1$ and $\sum_j |a_{ij}| = 1$ by (10).

The minimality of n implies $i \in J_i$ or equivalently, $a_{ii} \neq 0$: otherwise $a_i \equiv \pi(a_i) \equiv \sum_{j \in J_i \backslash \{i\}} a_{ij} a_j$ would be an $\mathbb{F}_\infty$-linear combination of the remaining columns of A.

For every $i \leq n$,

$$\begin{aligned} 1 &= \sum_j |b_{ij}| = \sum_j |\sum_k a_{ik} a_{kj}| \\ &\leq \sum_j \sum_k |a_{ik}||a_{kj}| = \sum_k |a_{ik}| \underbrace{\left(\sum_j |a_{kj}| \right)}_{=1} \\ &= 1, \end{aligned}$$

so equality holds. In particular, the terms $\operatorname{sgn}(a_{ik} a_{kj})$ are either all (for arbitrary $i, j, k \leq n$) non-negative or non-positive. Picking $k = j := i$, we see that they are non-negative, since $\operatorname{sgn}(a_{ii}^2) > 0$, for $a_{ii} \neq 0$.

Let $I^> := \{i, a_{ii} > 0\}$ and likewise with $I^<$. Then $I^> \sqcup I^- = \{1, \ldots, n\}$. Moreover, for $i \in I^>$ and $j \in I^<$, $a_{ii} a_{ij} \geq 0$ and $a_{ij} a_{jj} \geq 0$ imply $a_{ij} = 0$. In other words, the matrix A decomposes as a direct sum matrix $A^> \sqcup A^<$, where $A^>$ and $A^<$ are the submatrices of A consisting of the rows and columns with indices in $I^>$ and

$I^<$, respectively. We may therefore assume $A = A^>$, say. For $i (\in I^>)$, and any j, $a_{ii}a_{ij} \geq 0$ implies $a_{ij} \geq 0$, i.e., the entries of A are all non-negative.

Fix some $i \leq n$. As π is a projector, $a_i \equiv \pi(a_i)$, i.e.,

$$a_i \equiv \sum_{j \in J_i} a_{ij} e_j \equiv \sum a_{ij} \pi(e_j) \equiv \sum_{j \in J_i, k \in J_j} a_{ij} a_{jk} e_k \in \mathbb{F}_\infty(n).$$

By (10), (11), this implies $\mathrm{sgn}(a_{ik}) = \mathrm{sgn}(\sum_j a_{ij} a_{jk})$, which gives

$$J_i = \cup_{j \in J_i} J_j. \tag{12}$$

Indeed, "$\subset$" is easy to see without using the non-negativity of the entries. Conversely, for $k \notin J_i$, $\sum_j a_{ij} a_{jk} = 0$. Since all $a_{**} \geq 0$, this implies $a_{jk} = 0$ for all $j \in J_i$, i.e., $k \notin \cup_{j \in J_i} J_j$.

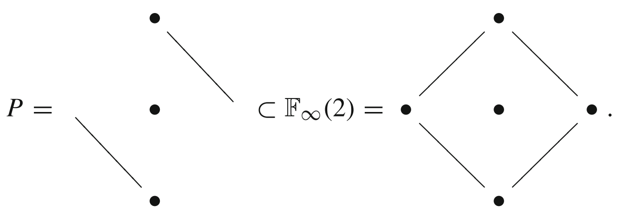
Now, pick some $i \leq n$ such that J_i is maximal, i.e., not contained in any other J_j, $i \neq j$. Then $i \notin J_j$ for any $i \neq j$ by (12). In other words, the i-th row only contains a single non-zero entry. For simplicity of notation, we may suppose $i = 1$.

Consider the diagram

$$\begin{array}{ccccc} \mathbb{F}_\infty & \overset{\iota}{\rightarrowtail} & \mathbb{F}_\infty(n) & \overset{\rho}{\twoheadrightarrow} & \mathbb{F}_\infty(n-1) \\ \| & & \downarrow & & \downarrow \\ \mathbb{F}_\infty & \rightarrowtail & M & \twoheadrightarrow & M' \end{array}$$

where ρ is the projection onto the last $n-1$ coordinates, ι is the injection in the first coordinate. The lower left-hand map is a monomorphism since the first row of A is nonzero. Its cokernel M' is the projective module determined by the matrix $(a_{ij})_{2 \leq i, j \leq n}$. This exact sequence shows that $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty)$ is generated by $[\mathbb{F}_\infty]$.

On the other hand, consider the projective $\mathbb{F}_\infty$-module P defined by the projector $\begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}$ [1, 10.4.20]. It consists of 5 elements and can be visualized as



The composition $\mathbb{F}_\infty \xrightarrow{(1/2,1/2)} \mathbb{F}_\infty(2) \twoheadrightarrow P$ is a monomorphism with cokernel $\mathbb{F}_\infty$. The pictured inclusion $P \to \mathbb{F}_\infty(2)$ has cokernel $\mathbb{F}_\infty$, spanned by e_1. This shows that $[\mathbb{F}_\infty(2)] = 2[\mathbb{F}_\infty] = [P] + [\mathbb{F}_\infty] = 3[\mathbb{F}_\infty]$. Hence $K_0^{\mathbf{Proj}}(\mathbb{F}_\infty) = 0$.

Finally, we have to show $K_0(\mathbb{F}_\infty) = \mathbb{Z}$. For this, consider a cofiber sequence

$$\mathbb{F}_\infty(n') \overset{i}{\rightarrowtail} \mathbb{F}_\infty(n) \overset{p}{\twoheadrightarrow} \mathbb{F}_\infty(n'').$$

We have to show $n = n' + n''$. Pick a section σ of p. The natural map $i \sqcup \sigma : \mathbb{F}_\infty(n') \sqcup \mathbb{F}_\infty(n'') \to \mathbb{F}_\infty(n)$ is injective, as one easily shows. Thus $n' + n'' \le n$ for cardinality reasons. Conversely, for any basis vector $e_i \in \mathbb{F}_\infty(n)\backslash \mathrm{im} i$, $p^{-1}(p(e_i)) = \{e_i\}$, as one shows in the same way as for $\mathbb{Z}_\infty$-modules, cf. (6). Thus $\sigma(p(e_i)) = e_i$, so there are at most n'' such basis vectors by the injectivity of σ. Moreover, at most n' of the basis vectors e_i of $\mathbb{F}_\infty(n)$ are in $\mathrm{im} i$ by the injectivity of i. This shows $n' + n'' \ge n$. □

Remark 4.3 For $p \le \infty$, let Fib be the homotopy fiber of $\Omega K(\mathbb{Z}_{(p)}) \to \Omega K(\mathbb{Q})$ and $\widehat{Fib}$ the one of $\Omega K(\mathbb{Z}_p) \to \Omega K(\mathbb{Q}_p)$. The localization sequence for K-theory shows in case $p < \infty$ that Fib and $\widehat{Fib}$ are homotopy equivalent (and given by $K(\mathbb{F}_p)$). Here Ω is the loop space and $K(-)$ is a space (or spectrum) computing K-theory, for example the $S_\bullet$-construction. However, for $p = \infty$, we have

$$\begin{array}{ccccccccc}
\pi_1(Fib) & \longrightarrow & K_1(\mathbb{Z}_{(\infty)}) & \longrightarrow & K_1(\mathbb{Q}) = \mathbb{Q}^\times & \longrightarrow & \pi_0(Fib) & \longrightarrow & 0 \\
 & & \| & & \downarrow \subsetneq & & & & \\
\pi_1(\widehat{Fib}) & \longrightarrow & \underbrace{K_1(\mathbb{Z}_\infty)}_{(\mathbb{Z}/2)^{\oplus 2}} & \longrightarrow & K_1(\mathbb{R}) = \mathbb{R}^\times & \longrightarrow & \pi_0(\widehat{Fib}) & \longrightarrow & 0,
\end{array}$$

so that $\pi_0(Fib) \subsetneq \pi_0(\widehat{Fib})$.

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