CONSTRUCTIBLE SHEAVES ON SCHEMES AND A CATEGORICAL KÜNNETH FORMULA
–
OR: "TWO SIMILARITIES BETWEEN ÉTALE SHEAVES AND QUASI-COHERENT SHEAVES"

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These are notes of a talk on the joint work with Tamir Hemo and Timo Richarz [HRS21b, HRS21a]. We refer to op. cit. for further references.

The talk has two goals:

(1) describe a robust formalism for constructible and lisse sheaves, for example ℓ-adic sheaves
(2) understand the relation between sheaves on two varieties $X_1, X_2/\mathbf{F}_p$ and their product $X_1 \times X_2$

1. Constructible sheaves

Recall from the work of Bhatt–Scholze that for a scheme X, the pro-étale site $X_{\text{proét}}$ consists of schemes Y/X that such that both the structural map $Y \to X$ and also the diagonal $Y \to Y \times_X Y$ are flat. Any étale map is pro-étale (in this case the second map is an open immersion). The pro-étale site is, however, decidedly larger than the étale site. For example, if $S = \lim S_i$ is a pro-finite set, then

$$X \times S = \lim(X \times S_i) \to X$$

is a pro-étale map. We denote the category of profinite sets, with their usual topology, by $*_{\text{proét}}$. A sheaf of commutative rings Λ on $*_{\text{proét}}$ is known as a condensed ring. The examples of condensed rings Λ relevant in this talk are discrete topological rings, and $\Lambda = \mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, $\bar{\mathbf{Q}}_\ell$. However, any topological ring, such as $\Lambda = \mathbf{R}$ and the adeles of a number ring, gives a condensed ring. We denote by $\mathrm{D}(X_{\text{proét}}, \Lambda)$ the (unbounded) derived (∞-)category of sheaves of Λ-modules, as introduced by Bhatt and Scholze. We refer to objects in that category as sheaves.

Definition 1.1. Let X be *any* scheme. A sheaf $M \in \mathrm{D}(X_{\text{proét}}, \Lambda)$ is called *lisse* if it is dualizable. We denote the corresponding subcategory of $\mathrm{D}(X_{\text{proét}}, \Lambda)$ by $\mathrm{D}_{\text{lis}}(X, \Lambda)$.

Recall that in a symmetric monoidal category C, an object c is dualizable if there is another object c' and coevaluation and evaluation maps $1 \stackrel{\text{coev}}{\to} c \otimes c'$, $c \otimes c' \stackrel{\text{ev}}{\to} 1$ such that the composites

$$c \stackrel{\text{id}\otimes\text{coev}}{\to} c \otimes c' \otimes c \stackrel{\text{ev}\otimes\text{id}}{\to} c$$

$$c' \stackrel{\text{id}\otimes\text{coev}}{\to} c' \otimes c \otimes c' \stackrel{\text{ev}\otimes\text{id}}{\to} c'$$

are the respective identity maps. For the purposes of this subject, one should think of dualizability as a finite-and-locally-constant condition:

- In the category of k-vector spaces, an object V is dualizable iff $\dim V < \infty$.
- In the category Mod_R of modules over some commutative ring R, an R-module is dualizable iff it is finitely generated projective.
- In the derived category $\mathrm{D}(\mathrm{Mod}_R)$, a complex is dualizable iff it is perfect, i.e., iff it is quasi-isomorphic to a bounded complex of finitely generated projective modules. We write Perf_R for the full subcategory of perfect complexes.

Lemma 1.2. *Let X be a w-contractible affine scheme (i.e., any pro-étale cover splits). Then there is an equivalence*

$$\mathrm{D}_{\text{lis}}(X, \Lambda) \cong \mathrm{Perf}_{\Gamma(X,\Lambda)}.$$

This is the first similarity of (lisse pro-)étale sheaves with quasi-coherent sheaves alluded to in the subtitle above. A main point of the pro-étale topology is that every scheme X admits a hypercovering by w-contractible affine schemes. This fact and the following lemma makes the category D_{lis} computable:

Lemma 1.3. *The functor $X \mapsto \mathrm{D}_{\text{lis}}(X, \Lambda)$ is a hypersheaf (of stable ∞-categories).*

Thus, even if X is an everyday geometric object, for example an algebraic variety, one can compute $\mathrm{D}_{\text{lis}}(X, \Lambda)$ by covering X with a lot of "dust", i.e., w-contractible schemes X'/X. Up on that dusty level, lisse sheaves are just perfect complexes, albeit over the large rings $\Gamma(X', \Lambda)$. A lisse sheaf is then a compatible collection of perfect complexes over $\Gamma(X', \Lambda)$, for all the w-contractibles X'/X. Note that such an approach requires having a notion of lisse sheaves on fairly general schemes, which did not exist in the literature before.

These facts allow one to describe $\mathrm{D}_{\mathrm{lis}}$ more concretely.

Theorem 1.4. *(1) If Λ is discrete, then a sheaf is lisse iff it is étale-locally a constant sheaf (associated to a perfect complex of Λ-modules).*

(2) If $\Lambda = \lim \Lambda_n$, a filtered limit with surjective maps and nilpotent kerneles, then $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) = \lim \mathrm{D}_{\mathrm{lis}}(X, \Lambda_n)$.

(3) Regarding localizations we have the following, where $\Lambda_* = \Lambda(*)$ denotes the underlying ring of the condensed ring Λ. For X quasi-compact and quasi-separated (qcqs), there is a fully faithful functor

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{Perf}_{T^{-1}\Lambda_*} \to \mathrm{D}_{\mathrm{lis}}(X, T^{-1}\Lambda).$$

(4) For a filtered colimit $\Lambda = \mathrm{colim}_i \Lambda_i$ of condensed rings, and X again being qcqs, there is an equivalence

$$\mathrm{D}_{\mathrm{lis}}(X, \mathrm{colim}\, \Lambda_i) = \mathrm{colim}\, \mathrm{D}_{\mathrm{lis}}(X, \Lambda_i).$$

(5) If X is topologically reasonably nice (e.g., irreducible), then a sheaf $M \in \mathrm{D}(X_{\mathrm{proét}}, \Lambda)$ is lisse iff it is pro-étale locally constant (again associated to a perfect complex of Λ_*-modules).

Proof. These facts all use the above hyperdescent statement, and then appropriate statements about perfect complexes. For example, the second statement reduces to a result of Bhatt, stating that $\mathrm{Perf}_R = \lim \mathrm{Perf}_{R_i}$, for $R = \lim R_i$ with surjective maps with nilpotent kernel. □

Once a well-behaved notion of lisse sheaves is in place, there is a standard recipe for setting up constructible sheaves: we define $\mathrm{D}_{\mathrm{cons}}(X, \Lambda) \subset \mathrm{D}(X_{\mathrm{proét}}, \Lambda)$ as the full subcategory consisting of sheaves M such that Zariski locally, on U_i, for $X = \bigcup U_i$, there is a finite covering $U_i = \bigcup_j U_{ij}$ by locally closed subschemes such that $M|_{U_{ij}}$ is lisse. All the statements above continue to hold for constructible (as opposed to lisse) sheaves. The one with localizations becomes even better: there is an equivalence

$$\mathrm{D}_{\mathrm{cons}}(X, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{Perf}_{T^{-1}\Lambda_*} \to \mathrm{D}_{\mathrm{cons}}(X, T^{-1}\Lambda).$$

The proof of this uses an arc descent statement due to Hansen–Scholze. For $\mathrm{D}_{\mathrm{lis}}$ instead of $\mathrm{D}_{\mathrm{cons}}$ the functor is generally not an equivalence.

Remark 1.5. It is worth pointing out that the categories $\mathrm{D}_{\mathrm{lis}}$ and $\mathrm{D}_{\mathrm{cons}}$ exist – in an a priori manner – for any condensed ring. The above theorem allows to *compute* these categories. This may be regard as the opposite approach to the classical one where one *defines*, say, $\mathrm{D}_{\mathrm{lis}}(X, \mathbf{Z}_\ell) := \lim \mathrm{D}_{\mathrm{lis}}(X, \mathbf{Z}/\ell^n)$.

2. A categorical Künneth formula for constructible Weil sheaves

If X_1, X_2 are varieties over an algebraically closed field k, and F_1, F_2 are two constructible $\mathbf{Q}_\ell$-adic sheaves on them, the the classical Künneth formula asserts an isomorphism

$$\bigoplus_{a+b=n} \mathrm{H}^a(X_1, F_1) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^b(X_2, F_2) \stackrel{\cong}{\to} \mathrm{H}^n(X_1 \times_k X_2, F_1 \boxtimes F_2).$$

Here $F_1 \boxtimes F_2$ is the exterior product (for example, ${\mathbf{Q}_\ell}_{X_1} \boxtimes {\mathbf{Q}_\ell}_{X_2} = {\mathbf{Q}_\ell}_{X_1 \times X_2}$). With little effort, this formula can be recast as the *full faithfulness* of the functor

$$\mathrm{D}_{\mathrm{cons}}(X_1) \otimes_{\mathrm{Perf}_{\mathbf{Q}_\ell}} \mathrm{D}_{\mathrm{cons}}(X_2) \stackrel{\boxtimes}{\to} \mathrm{D}_{\mathrm{cons}}(X_1 \times X_2).$$

Here, the $\otimes$ is the tensor product, due to Lurie, of stable idempotent complete ∞-categories. Note that $\mathrm{Perf}_{\mathbf{Q}_\ell} = \mathrm{D}_{\mathrm{cons}}(k)$, so that the functor in question is about the behaviour of $\mathrm{D}_{\mathrm{cons}}$ with respect to products.

The similar functor for QCoh, the (stable presentable ∞-)category $\mathrm{QCoh}(-)$ of (complexes of) quasi-coherent sheaves is known to be an equivalence:

$$\mathrm{QCoh}(X_1) \otimes_{\mathrm{D}(\mathrm{Mod}_k)} \mathrm{QCoh}(X_2) = \mathrm{QCoh}(X_1 \times_k X_2).$$

For $\mathrm{D}_{\mathrm{lis}}$, though, the above functor fails to be an equivalence. This has two reasons:

- If a sheaf F on $X_1 \times X_2$ is of the form $F_1 \boxtimes F_2$, then its support is essentially of the form $Z_1 \times Z_2$, for $Z_i \subset X_i$. But it is easy to come up with sheaves having different support, e.g., $\Delta_* \mathbf{Q}_\ell$, for $\Delta : \mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{A}^1$ the diagonal.
- A more subtle reason why it fails occurs for schemes in characteristic p. The natural map

  $$\pi_1(X_1 \times_k X_2) \to \pi_1(X_1) \times \pi_1(X_2)$$

  is known to be an isomorphism if $\mathrm{char}\, k = 0$, and also in positive characteristic if X_1 or X_2 is proper. However, for $X_i = \mathbf{A}^1_{\overline{\mathbf{F}}_p}$, the map is surjective, but not injective. The reason for this are Artin-Schreier-type coverings

  $$U = \{(x, y, t) \mid t^p - t = xy\} \to \mathbf{A}^2.$$

Such U has non-isomorphic fibers over various $x \times \mathbf{A}^1$, and therefore does not decompose as $U = U_1 \times U_2$ for finite etale covers U_i/X_i.

The solution to both issues are Weil sheaves. For a scheme $X/\mathbf{F}_p$, let $\overline{X} := X \times_{\mathbf{F}_p} \overline{\mathbf{F}}_p$ and $\phi_X := \mathrm{id}_X \times \mathrm{Frob}_{\overline{\mathbf{F}}_p} : \overline{X} \to \overline{X}$. In terms of coordinates, if $(x_1, \dots, x_n)$ is a point in $\overline{X}$, then ϕ_X maps it to (x_i^p).

Definition 2.1. (Deligne) A Weil sheaf on a scheme $X/\mathbf{F}_p$ is a pair

$$(M \in \mathrm{D}_{\mathrm{cons}}(\overline{X}, \mathbf{Q}_\ell), \alpha : M \overset{\cong}{\to} \phi_X^* M).$$

More succinctly, we define

$$\mathrm{D}(X^{\mathrm{Weil}}) := \lim(\mathrm{D}_{\mathrm{cons}}(X, \mathbf{Q}_\ell) \underset{\mathrm{id}}{\overset{\phi_X^*}{\rightrightarrows}} \mathrm{D}_{\mathrm{cons}}(X, \mathbf{Q}_\ell)),$$

and then the objects in that category are the above pairs.

Lemma 2.2. *(essentially due to Geisser) If we replace $\mathbf{Q}_\ell$ by a finite (discrete) ring above, then $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda) = \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$.*

That is, for these rings the passage to Weil sheaves makes no difference to ordinary étale constructible sheaves. However, for other coefficients these do differ.

Let us see how the passage to Weil sheaves affects the above question. The issue with supports is (after some reduction steps) treated by the following lemma.

Lemma 2.3. *If $X/\mathbf{F}_p$ is qcqs, and $k/\mathbf{F}_p$ is a separably closed extension field (the point of interest being not $k = \overline{\mathbf{F}}_p$, but rather $k = \overline{\mathbf{F}_p(t)}$, $k = \overline{k(X_2)}$ etc.!), then there is a bijection between*

- *constructible subsets in X,*
- *constructible subsets $Z \subset X \times \operatorname{Spec} k$ such that set-theoretically $\mathrm{Frob}_X(Z) = Z$. Here $\mathrm{Frob}_X = \mathrm{Frob}_X \times \mathrm{id}_k$; equivalently one may also take $\mathrm{id}_X \times \mathrm{Frob}_k$ instead.*

Proof. The proof of this uses a lemma of Drinfeld–Lang: for $X/\mathbf{F}_p$ projective there is an equivalence between the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X and the category $\lim(\mathrm{Coh}(X \times k) \underset{\mathrm{id}}{\overset{\mathrm{Frob}_k^*}{\rightrightarrows}} \mathrm{Coh}(X \times k))$. ☐

The issue with the non-compatibility of π_1 with products is also fixed by the passage to Weil sheaves. This insight is due to Drinfeld. Recall that for X geometrically connected, there is a pair of short exact sequences

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(X_{\overline{\mathbf{F}}_p}) & \to & \mathrm{Weil}(X) & \to & \mathrm{Weil}(\mathbf{F}_p) = \langle \mathrm{Frob}_{\overline{\mathbf{F}}_p} \rangle \cong \mathbf{Z} & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1(X_{\overline{\mathbf{F}}_p}) & \to & \pi_1(X) & \to & \mathrm{Gal}(\mathbf{F}_p) \cong \hat{\mathbf{Z}} & \to & 1. \end{array}$$

More succinctly, one can define the Weil group*oid* as the quotient (in groupoids)

$$\mathrm{Weil}(X) = \Pi_1(X \times \overline{\mathbf{F}}_p)/\mathrm{Frob}_{\overline{\mathbf{F}}_p}.$$

If X is geometrically connected, then this is the groupoid to the Weil group mentioned above.

Now, for $X_1, X_2/\mathbf{F}_p$, the Frobenius–Weil groupoid (appearing, for example, in the work by Vincent Lafforgue) is defined as

$$\mathrm{FWeil}(X_1, X_2) := \Pi_1(X_1 \times X_2 \times \overline{\mathbf{F}}_p)/\mathrm{Frob}_{X_1}, \mathrm{Frob}_{X_2}.$$

Again, if $X := X_1 \times X_2$ is geometrically connected, this is the groupoid associated to the group appearing in an exact sequence

$$1 \to \pi_1(X \times \overline{\mathbf{F}}_p) \to \mathrm{FWeil}(X_1, X_2) \to \mathbf{Z}^2 \to 1.$$

The key insight, essentially due to Drinfeld (with an addendum for $\mathbf{Q}_\ell$-coefficients due to Xue) is: for two algebraic varieties $X_1, X_2/\mathbf{F}_p$, there is an equivalence

$$\mathrm{Rep}_{\mathbf{Q}_\ell}(\mathrm{Weil}(X_1) \times \mathrm{Weil}(X_2)) \overset{\cong}{\to} \mathrm{Rep}_{\mathbf{Q}_\ell}(\mathrm{FWeil}(X_1, X_2)).$$

One uses this and the fact that representations of the Weil group(oids) generate $\mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}) \subset \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}})$ in order to prove:

Theorem 2.4. *For two algebraic varieties $X_1, X_2/\mathbf{F}_p$, there is an equivalence given by $\boxtimes$:*

$$\mathrm{D}(X_1^{\mathrm{Weil}}, \mathbf{Q}_\ell) \otimes_{\mathrm{Perf}_{\mathbf{Q}_\ell}} \mathrm{D}(X_2^{\mathrm{Weil}}, \mathbf{Q}_\ell) \cong \mathrm{D}(X_1^{\mathrm{Weil}} \times X_2^{\mathrm{Weil}}, \mathbf{Q}_\ell),$$

where the right hand side is defined as the homotopy fixed points of the two commuting Frob_{X_i}-pullback functors.

Given the product formula $\mathrm{QCoh}(X_1) \otimes_{\mathrm{D}(\mathrm{Mod}_k)} \mathrm{QCoh}(X_2) = \mathrm{QCoh}(X_1 \times_k X_2)$, this gives a second similarity of (Weil) sheaves with quasi-coherent sheaves.

References

[HRS21a] Tamir Hemo, Timo Richarz, and Jakob Scholbach. A categorical Künneth formula for Weil sheaves, 2021. arXiv:2012.02853.

[HRS21b] Tamir Hemo, Timo Richarz, and Jakob Scholbach. Constructible sheaves on schemes, 2021. arXiv:2305.18131.