Homology and Cohomology

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Chapter 0

Preface

These are notes for a lecture on algebraic topology, more specifically basic notions around homology and cohomology, offered in Spring 2022 and Spring 2023 at the University of Padova.

The exclamation mark (!)indicates that you should repeat some aspects of a definition etc. in order to make sure you are following the lecture.

Chapter 1

Introduction

Homology and *Cohomology* are fundamental techniques in algebraic topology. By its nature, topology is a very flexible subject – think of continuously deforming a cup into a doughnut. The reason for this flexibility is that there is often an abundance of continuous maps between two topological spaces. This holds even for very "standard" spaces. Therefore, questions such as the following are not altogether trivial to answer:

Question 1.1. Suppose $n, m \in \mathbf{N}$ with $n < m$. Is there a continuous surjective map

$$\mathbf{R}^n \to \mathbf{R}^m?$$

Is there a homeomorphism

$$\mathbf{R}^n \to \mathbf{R}^m?$$

The answer to the first question is yes (cf. Exercise 1.2), which shows how limited our intuition about topological spaces really is. Second, it will take us some time to prove (see below) that the answer to the second question is no.

By comparison, other mathematical areas offer less freedom, so the following lemma from linear algebra is by comparison decidely easier than the above:

Lemma 1.2. Again for $n, m \in \mathbf{N}$ with $n < m$ there is no $\mathbf{R}$-linear surjective map

$$\mathbf{R}^n \to \mathbf{R}^m$$

and, in particular, no $\mathbf{R}$-linear isomorphism.

The overall workflow of algebraic topology is this:

- Take a question about topological or geometrical objects, e.g.: "are two topological spaces X and Y homeomorphic?"
- Convert this hard question into a much easier question about objects in linear algebra, such as "are two vector spaces V and W isomorphic"?
- Transfer information from the linear algebraic objects to the topological objects.

This workflow can be implemented in different ways. In this course, we will focus on homology and cohomology, which are the easiest ways of converting topological information into linear algebraic information.

1.1 The Eilenberg–Steenrod axioms

Homology has the following properties, known as *Eilenberg–Steenrod axioms*. At this point, we state them in a slightly simplified form. Proving this theorem will keep us busy for a good while.

Theorem 1.3. (1) (*Functoriality*) Homology is a functor

$$\mathrm{H}_n : \mathrm{Top} \to \mathrm{Ab},\ n \geqslant 0.$$

That is, for each topological space X, there is an abelian group

$$\mathrm{H}_n(X).$$

In addition, for each continuous map $f : X \to Y$ there is a group homomorphism

$$\mathrm{H}_n(f) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y).$$

Functoriality means that these group homomorphisms have the following key property: for another continuous map $g : Y \to Z$, the map $\mathrm{H}_n(X) \stackrel{\mathrm{H}_n(f)}{\to} \mathrm{H}_n(Y) \stackrel{\mathrm{H}_n(g)}{\to} \mathrm{H}_n(Z)$ agrees with the map $\mathrm{H}_n(X) \stackrel{\mathrm{H}_n(g\circ f)}{\to} \mathrm{H}_n(Z)$. More succinctly:

$$\mathrm{H}_n(g \circ f) = \mathrm{H}_n(g) \circ \mathrm{H}_n(f) \qquad (1.4).$$

In addition,

$$\mathrm{H}_n(\mathrm{id}_X) = \mathrm{id}_{\mathrm{H}_n(X)} : \mathrm{H}_n(X) \to \mathrm{H}_n(X). \qquad (1.5)$$

(2) (*Dimension axiom*, cf. Proposition 4.3) The homology of a point is this:

$$\mathrm{H}_n(\{*\}) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

(3) (*Additivity*, cf. Proposition 4.4) Homology of a disjoint union of spaces is the direct sum of the homologies of the individual spaces:

$$\mathrm{H}_n\left(\bigsqcup_{i \in I} X_i\right) = \bigoplus_{i \in I} \mathrm{H}_n(X_i).$$

(4) (*Homotopy invariance*, cf. Proposition 4.8) If two continuous maps $f, g : X \to Y$ are homotopic (i.e., there is a continuous family of maps $h_t : X \to Y$ for $t \in [0,1]$ such that $h_0 = f$, $h_1 = g$), then the induced maps on homology

$$\mathrm{H}_n(f), \mathrm{H}_n(g) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

are the same maps.

(5) (*Excision* or *Mayer–Vietoris sequence*, Corollary 4.20) If $U, V \subset X$ are two subspaces such that X is covered by the interiors of U, V,

$$X = U^\circ \cup V^\circ,$$

then there is a long exact sequence

$$\dots \to \mathrm{H}_n(U \cap V) \to \mathrm{H}_n(U) \oplus \mathrm{H}_n(V) \to \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(U \cap V) \to \dots \blacksquare$$

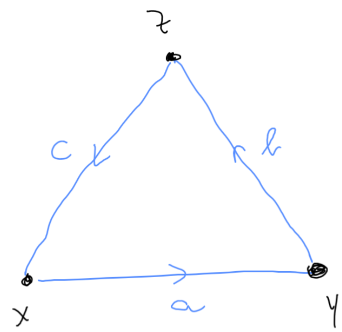
All the terms mentioned in this theorem will be explained during the course. At this point, let us just weigh the nature of these statements:

- The definition of the homology functors is a sequence of functors

  $$\mathrm{Top} \stackrel{\mathrm{Sing}}{\to} \mathrm{sSet} \stackrel{\mathbf{Z}[-]}{\to} \mathrm{sAb} \stackrel{N}{\to} \mathrm{Ch} \stackrel{\mathrm{H}_n}{\to} \mathrm{Ab}.$$

  where sSet is the category of *simplicial sets*, which provide a means to control the combinatorics in a space equipped with a triangulation.

  For $S^1 \in \mathrm{Top}$, the first three functors together do *roughly* the following: we pretend that S^1 can be replaced by the following "space" "S^1"



This "space" "S^1" has 3-zero dimensional points x, y, z, as well as 3 1-dimensional edges a, b, c. Each edge has two endpoints, denoted by d_0 and d_1, namely

	d_0	d_1
a	y	x
b	z	y
c	x	z.

We associate to this combinatorial datum two rank-3 abelian groups, and a map between them

$$\mathbf{Z}a \oplus \mathbf{Z}b \oplus \mathbf{Z}c \xrightarrow{\partial} \mathbf{Z}x \oplus \mathbf{Z}y \oplus \mathbf{Z}z.$$

The so-called differential ∂ sends an edge e to $d_0(e) - d_1(e)$, e.g., $a \mapsto y - x$ etc. Thus, ∂ is described by the matrix

$$\begin{pmatrix} -1 & & 1 \\ 1 & -1 & \\ & 1 & -1 \end{pmatrix}.$$

The homology groups

$$\mathrm{H}_0(\text{“}S^1\text{”}), \mathrm{H}_1(\text{“}S^1\text{”})$$

are defined to be the cokernel, resp. the kernel of ∂. As it is, the cokernel is(!)a free abelian group of rank one, i.e., isomorphic to $\mathbf{Z}$, e.g., generated by $[x](= [y] = [z])$. The kernel is also a free abelian group of rank one, generated by $a + b + c$.

We obtain the result

$$\mathrm{H}_n(\text{“}S^1\text{”}) = \begin{cases} \mathbf{Z}x & n = 0 \\ \mathbf{Z}(a+b+c) & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

The "space" "S^1" will be an example of a simplicial set. The functor Sing sends S^1 to a simplicial set whose points (a.k.a. 0-simplices) are just the points in S^1, and whose edges (a.k.a. 1-simplicse) are continuous maps $[0,1] \to S^1$. Thus, $\mathrm{Sing}(S^1)$ is much larger than "S^1". It will turn out, however, that

$$\mathrm{H}_n(S^1) = \mathrm{H}_n(\text{“}S^1\text{”}),$$

i.e., the difference between $\mathrm{Sing}(S^1)$ and "S^1" is, however, negligible as far as the end result of the computation, the homology groups, are concerned.

- The proof of the dimension axiom and additivity is elementary.

- Homotopy invariance allows us to compute, say, the homology of any convex set $\varnothing \neq X \subset \mathbf{R}^n$. Indeed, homotopy invariance quickly implies

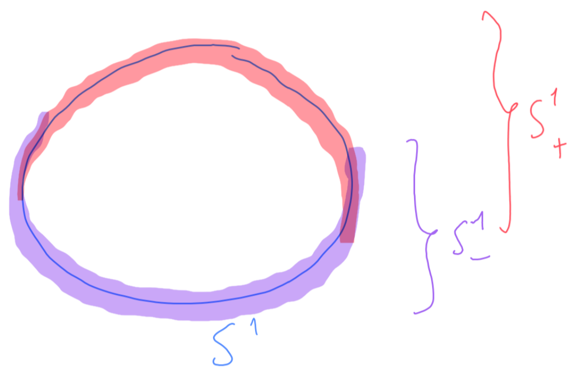
$$\mathrm{H}_*(X) = \mathrm{H}_*(\{*\}). \tag{1.6}$$

The proof of the homotopy invariance is less immediate than the previous ones, but is a beautiful showcase for the appeal of structure-based mathematics. We will show that homotopies in Top are mapped to a closely related notion of homotopies in the category sSet of simplicial sets, which in turn are mapped to chain homotopies of chain complexes. Finally, taking homology then produces identical maps. Neither of these steps is particularly difficult.

- The excision axiom or Mayer–Vietoris sequence is the key tool for computing homology of non-trivial spaces such as S^n. For example, for the covering

$$S^1 = S^1_+ \cup S^1_-$$

as depicted



we have, by homotopy invariance

$$\mathrm{H}^n(S^1_\pm) = \mathrm{H}^n(\{*\}) = \mathbf{Z} \text{ for } n = 0, 0 \text{ otherwise.}$$

Also, again by homotopy invariance and additivity,

$$\mathrm{H}_n(S^1_+ \cap S^1_-) = \mathrm{H}_n(\{*\}) \oplus \mathrm{H}_n(\{*\}) = \mathbf{Z} \oplus \mathbf{Z} \text{ for } n = 0, 0 \text{ otherwise .}$$

Thus, the above-mentioned long exact sequence becomes

$$0 \to \mathrm{H}_1(S^1) \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z} \to \mathrm{H}_0(S^1) \to 0.$$

As we will see, the map in the middle is given by $f : (x, y) \mapsto (x + y, x + y)$, whose kernel is

$$\mathrm{H}_1(S^1) = \ker f \cong \mathbf{Z},$$

$$\mathrm{H}_0(S^1) = \operatorname{coker} f \cong \mathbf{Z}.$$

The end result of this computation is

$$\mathrm{H}_n(S^1) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Using similar arguments, we will prove in Proposition 4.21, for $k > 0$:

$$\mathrm{H}_n(S^k) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z} & n = k \\ 0 & \text{otherwise} \end{cases}$$

The intuition behind homology is that the n-th homology counts the number of n-dimensional holes. For S^n, there is precisely one such hole, in line with $\mathrm{H}_n(S^n) = \mathbf{Z}^1$.

1.2 First Applications

Let

$$D^n := \{(x_1, \dots, x_n) \in \mathbf{R}^n, \sum x_k^2 \leqslant 1\}$$

be the n-dimensional *ball* with radius 1. Its boundary

$$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbf{R}^n, \sum x_k^2 = 1\}$$

is the $n-1$-dimensional *sphere*.

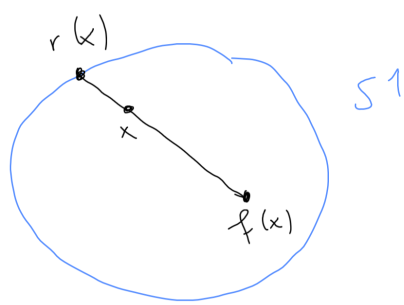
Theorem 1.7. (*Brouwer fixed point theorem*) Let $n \geqslant 0$ and $f : D^{n+1} \to D^{n+1}$ be a continuous map. Then f has a fixed point, i.e., there is some $x \in D^{n+1}$ with

$$f(x) = x.$$

Proof. Suppose f has no fixed point. Then the ray starting at $f(x)$ and passing through x intersects S^n in exactly one point, denoted $r(x)$. One shows that the function

$$r : D^{n+1} \to S^n$$

is continuous.



For $x \in S^n$ we clearly have $r(x) = x$. In other words, writing $i : S^n \to D^{n+1}$ for the inclusion, we have

$$r \circ i = \mathrm{id}_{S^n}.$$

With these preliminaries, we can make use of the above Eilenberg–Steenrod axioms (which we prove later). By functoriality, the induced map on the n-th homology groups read

$$\mathbf{Z} = \mathrm{H}_n(S^n) \stackrel{\mathrm{H}_n(i)}{\to} 0 = \mathrm{H}_n(D^{n+1}) \stackrel{\mathrm{H}_n(r)}{\to} \mathrm{H}_n(S^n).$$

Thus, the composite must be the zero map. On the other hand, by the functoriality of homology, we have

$$\mathrm{H}_n(r) \circ \mathrm{H}_n(i) = \mathrm{H}_n(r \circ i) = \mathrm{H}_n(\mathrm{id}_{S^n}) = \mathrm{id}_{\mathrm{H}_n(S^n)} = \mathrm{id}_{\mathbf{Z}}.$$

We obtain a contradiction, since certainly the identity map of $\mathbf{Z}$ is not the same as the zero map. $\square$

Theorem 1.8. (*Topological invariance of dimension*) There is a homeomorphism

$$S^n \to S^m,$$
$$\mathbf{R}^n \to \mathbf{R}^m$$

(if and) only if $n = m$.

Proof. If $f : S^n \to S^m$ is a homeomorphism with (continuous) inverse g, then

$$f \circ g = \mathrm{id}, g \circ f = \mathrm{id}.$$

Again using (1.4) and (1.5), this implies

$$\mathrm{H}_n(f) \circ \mathrm{H}_n(g) = \mathrm{id}$$

and similarly the other way round. That is, $\mathrm{H}_n(f)$ is an isomorphism. If $n = 0$, S^0 consists of two points and is not even bijective to S^m for $m > 0$. We may thus assume $n \geqslant 1$. For $n \neq m$, however, $\mathrm{H}_n(S^m) = 0$ is *not* isomorphic to $\mathrm{H}_n(S^n) = \mathbf{Z}$. Note how much easier it is to decide whether

$$0 \stackrel{?}{\cong} \mathbf{Z}$$

than to decide

$$\mathbf{R}^n \stackrel{?}{\cong} \mathbf{R}^m.$$

Here, it is simple: 0 is finite, while $\mathbf{Z}$ is not. Alternatively, one may compare the ranks of these two groups, which are 0 and 1, respectively, so the groups are not isomorphic.

If $f : \mathbf{R}^n \to \mathbf{R}^m$ is a homeomorphism, then there is also a homeomorphism

$$f : \mathbf{R}^n \backslash \{0\} \to \mathbf{R}^m \backslash \{f(0)\}.$$

We may assume $n \geqslant 1$ since a homeomorphism $\mathbf{R}^0 \to \mathbf{R}^m$ is in particular a bijection, so that $m = 0$. For $n \geqslant 1$, the inclusion $S^{n-1} \subset \mathbf{R}^n \backslash \{0\}$ is such that there is (!)a map (called retraction)

$$r : \mathbf{R}^n \backslash \{0\} \to S^{n-1}$$

such that $r \circ i = \mathrm{id}$ and $i \circ r$ is homotopic to (but not the same as) $\mathrm{id}_{\mathbf{R}^n \backslash \{0\}}$. The homotopy invariance axiom and functoriality of H_n thus imply that $\mathrm{H}_n(i)$ is an isomorphism (with $\mathrm{H}_n(r)$ being its inverse). Thus

$$\mathrm{H}_*(\mathbf{R}^n \backslash \{0\}) = \mathrm{H}_*(S^{n-1}).$$

Similarly (use translation by $f(0)$) for $\mathbf{R}^m \backslash \{f(0)\}$. Then the same argument as before applies. □

A more refined study of the homology of spheres yields further applications:

- The map $S^1 \to S^1 (:= \{z \in \mathbf{C}, |z| = 1\})$, $z \mapsto z^n$ induces a map

  $$\mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1).$$

  Using the above computation, this tranlates into a map

  $$\mathbf{Z} \to \mathbf{Z}$$

  which we will show to be the multiplication by n. This insight, together with homotopy invariance, can be used to prove the fundamental theorem of algebra (every nonconstant complex polynomial has a root), cf. Corollary 4.41.

- The hedgehog theorem (Corollary 4.40) states that there is no non-zero tangent vector field at even-dimensional spheres.

Beyond these classical applications within geometry, homology and cohomology are also omnipresent in other areas. A rather new development, known as *persistent homology*, aims to use homology in order to exhibit patterns in high-dimensional datasets, such as those occurring in analysis of medical images.

The ideas and methods encountered in this lecture also inform other areas such as algebraic geometry, which often draws inspiration from the intuition gained by results such as the ones presented in this course.

1.3 Exercises

Exercise 1.1. Define a slight modification of the "space" "S^1", to be denoted "D^2", and consider three abelian groups

$$\mathbf{Z} \xrightarrow{\partial_2} \mathbf{Z}a \oplus \mathbf{Z}b \oplus \mathbf{Z}c \xrightarrow{\partial_1} \mathbf{Z}x \oplus \mathbf{Z}y \oplus \mathbf{Z}z.$$

What map ∂_2 reflects the geometric intuition that the boundary of the interior of a triangle consists of the edges a, b, and c? Compute

$$\begin{aligned} \mathrm{H}_2(\text{“}D^2\text{”}) &:= \ker \partial_2 \\ \mathrm{H}_1(\text{“}D^2\text{”}) &:= \ker \partial_1 / \mathrm{im}\ \partial_2 \\ \mathrm{H}_0(\text{“}D^2\text{”}) &:= \operatorname{coker} \partial_1. \end{aligned}$$

If you have chosen the right map ∂_2, the groups $\mathrm{H}_n(\text{“}D^2\text{”})$ you obtain will be the same as the groups $\mathrm{H}_n(D^2)$, which are $\mathbf{Z}$ for $n = 0$ and 0 for $n > 0$ (cf. (1.6)).

Exercise 1.2. In a textbook of your choice, read about the *Peano curve*, a continuous surjective map

$$[0, 1] \to [0, 1] \times [0, 1].$$

Exercise 1.3. Let X be a topological space. Consider the following relation $\sim$ on X:

$$x \sim y$$

if and only if there is a continuous map $h : [0, 1] \to X$ with $h(0) = x, h(1) = y$. Prove:

- $\sim$ is an equivalence relation. (Hint: for one property you will need to use a lemma from elementary topology.)
- We define the *set of path components* $\pi_0(X) := X/\sim$.
- Compute $\pi_0([0, 1])$ and $\pi_0(\mathbf{Z})$ (here $\mathbf{Z} \subset \mathbf{R}$ has its usual discrete topology).
- We call X *path-connected* if $\pi_0(X)$ has at most one element. Let $f : X \to Y$ be a continuous surjective map. Show that Y is path-connected if X is so.

Chapter 2

Simplicial sets

Simplicial sets form the technical backbone of algebraic topology, and are also of paramount importance in contemporary higher categorical structures such as ∞-categories. In this section, we develop the basics of this concept. References for the material in this section include [May92; GJ09; Fri08; Lur].

2.1 Definitions

Simplicial sets are supposed to provide a combinatorial model for topological spaces equipped with a triangulation. We begin with the (quite abstract) definition, and will gradually gain a more geometric understanding of this notion.

Definition 2.1. Let C be a category, e.g. $C = \mathrm{Set}$. A *semi-simplicial object* in C is a sequence of objects $X_n \in C$ ($n \geqslant 0$), together with maps

$$d_k : X_n \to X_{n-1}, 0 \leqslant k \leqslant n,$$

called *face maps*, such that

$$d_i \circ d_j = d_{j-1} \circ d_i \tag{2.2}$$

for $i < j$ (for any n, note both composites are maps $X_n \to X_{n-2}$). (Strictly speaking, d_k is an abuse of notation, a more complete notation would be $d_k^n : X_n \to X_{n-1}$, but we stick to that reduced notation.)

A *simplicial object* in C is a sequence of object $X_n \in C$, face maps d_k as above, and in addition *degeneracy maps*

$$s_k : X_n \to X_{n+1}, 0 \leqslant k \leqslant n,$$

such that the following *simplicial identities* hold

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, \text{ for all } i < j, \\ s_i s_j &= s_j s_{i-1}, \text{ for all } i > j, \\ d_i s_j &= \begin{cases} s_{j-1} d_i & i < j \\ \text{id} & i = j \text{ or } i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases} \end{aligned} \tag{2.3}$$

For C = Set, the elements of X_n are called n-*simplices* of X. 0-simplices are also called *vertices*, 1-simplices are *edges*. An n-simplex is called *degenerate* if it is in the image of some degeneracy map s_i.

Example 2.4. We define a simplicial set Δ^0 to be such that $(\Delta^0)_n = \{*\}$ and such that all face and degeneracy maps are the identity. More generally, for any set X, there is a simplicial set disc (X) (or just also denoted by X again), the *discrete simplicial set* associated to X, given by

$$\text{disc } (X)_n := X$$

and all maps d_k and s_k are id_X. Thus $\Delta^0 = \text{disc } (\{*\})$.

Example 2.5. Moving up in dimension 1, we define a semi-simplicial set $\tilde{\Delta}^1$ by

$$(\tilde{\Delta}^1)_1 := \{01\}, (\tilde{\Delta}^1)_0 := \{0, 1\}.$$

Here 01 is just a symbol that serves as a mnemonic for a line going from 0 to 1. In line with this, we let

$$d_0(01) := 1, d_1(01) := 0. \tag{2.6}$$

The purpose of d_k is to remember that the endpoints of the line 01 are 0 and 1, respectively, with the idea that 1 is the endpoint *opposite* to 0 (so that $d_0(01) = 1$, as opposed to 0). We finally define $(\tilde{\Delta}^1)_n := \varnothing$ for $n \geqslant 2$. This defines a semi-simplicial set.

This semi-simplicial set $\tilde{\Delta}^1$ is *not* a simplicial set: the only way we could define $s_0 : (\tilde{\Delta}^1)_0 \to (\tilde{\Delta}^1)_1$ is to send 0 and 1 to 01, which would violate the simplicial identity $d_1 s_0(0) \stackrel{!}{=} 0$. The way out of this is to enlarge $\tilde{\Delta}^1$ as follows: we define

$$(\Delta^1)_1 := \{00, 01, 11\} \to (\Delta^0)_0 := \{0, 1\}.$$

Again, 00 etc. are just formal symbols, with the idea that the symbol ij represents a line from i to j, so that 00 is a "constant" line at 0. Extending the above, we define

$$d_k : (\Delta^1)_1 \to (\Delta^1)_0, d_0(i_0 i_1) := i_1, d_1(i_0 i_1) := i_0.$$
$$s_0 : (\Delta^1)_0 \to (\Delta^1)_1, s_k(i_0) = i_0 i_0.$$

(To memorize the definition of d_k: d_k removes the k-th entry.) We also need to specify 2-simplices. Unlike for $\tilde{\Delta}^1$, we cannot define Δ^1_2 to be empty, since we need to supply $s_0, s_1 : \Delta^1_1 \to \Delta^1_2$. We will shortly complete the definition of Δ^1.

Example 2.7. We define a semi-simplicial set $\tilde{S}^1$ (resp. a simplicial set S^1) in low degrees by

$$(\tilde{S}^1)_1 := \{\gamma\} \stackrel{d_0, d_1}{\to} (\tilde{S}^1)_1 := \{*\}$$

Again, to define a simplicial set, one needs to enlarge the set $\{01\}$ slightly, so that we put

$$(S^1)_1 := \{\gamma, *\} \stackrel{d_0, d_1}{\to} (S^1)_1 := \{*\}$$

with $s_0(*) = *$. The maps d_0 and d_1 encode the idea that γ is a closed *loop*, i.e., a path whose two endpoints are the same.

In order to conveniently complete the definition of Δ^1 (and later also S^1), including all the higher-dimensional simplices, we use the following category.

Definition 2.8. The *simplex category* Δ has objects

$$[n] = \{0, 1, \ldots, n\}$$

for $n \geqslant 0$, and morphisms are *order-preserving maps*:

$$\mathrm{Hom}_\Delta([m], [n]) = \{\alpha : [m] \to [n], \alpha(i) \leqslant \alpha(j) \text{ for all } i \leqslant j\}.$$

In this category, there are the following important morphisms:

$$\delta_k : [n-1] \to [n], 0 \leqslant k \leqslant n$$

is the unique injective map that misses $k \in [n]$. Somewhat dually,

$$\sigma_k : [n+1] \to [n], 0 \leqslant k \leqslant n$$

is the unique surjective map that repeats k (i.e., maps $0, 1, \dots, n, n+1$ to $0, 1, \dots, k, k, \dots n$). We also consider the category $\Delta_{\text{inj}} \subset \Delta$ having the same objects, but only those morphisms that are injective (equivalently, strictly increasing).

Definition and Lemma 2.9. For $e \geqslant 0$, we define

$$(\Delta^e)_n := \operatorname{Hom}_\Delta([n], [e]).$$

We define the face maps to be

$$d_k : (\Delta^e)_n = \operatorname{Hom}_\Delta([n], [e]) \to (\Delta^e)_{n-1} = \operatorname{Hom}_\Delta([n-1], [e])$$

to be the precomposition with δ_k, i.e., $f : [n] \to [e]$ is mapped to the composition

$$[n-1] \xrightarrow{\delta_k} [n] \xrightarrow{f} [e].$$

We also define the face maps similarly:

$$s_k : (\Delta^e)_n = \operatorname{Hom}_\Delta([n], [e]) \to (\Delta^e)_{n+1} = \operatorname{Hom}_\Delta([n+1], [e])$$

by

$$s_k(f) := f \circ \sigma_k.$$

This defines a simplicial set Δ^e, called the *e-simplex*.

Example 2.10. We have

$$(\Delta^1)_0 = \operatorname{Hom}_\Delta([0], [1]) = \{0, 1\},$$
$$(\Delta^1)_1 = \operatorname{Hom}_\Delta([1], [1]) = \{00, 01, 11\},$$

where ij is a shorthand for the map $0 \mapsto i$, $1 \mapsto j$. This recovers Example 2.5:

$$s_0 : \{0, 1\} \to \{00, 01, 11\}$$

sends $i (= 0, 1)$ to the map

$$[1] \xrightarrow{\sigma_0} [0] \xrightarrow{i} [1],$$

which is just ii. Similarly d_k (for $k = 0, 1$) sends ij to the composite

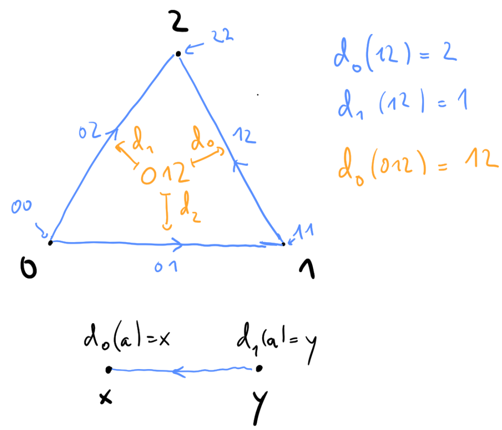
$$[0] \xrightarrow{\delta_k} [1] \xrightarrow{ij} [1].$$

Since δ_k *misses* k, $\delta_0(0) = 1$, so that $d_0(ij) = j$, while $d_1(ij) = i$. This recovers (2.6). We then have, using similar notation,

$$(\Delta^1)_2 = \mathrm{Hom}_\Delta([2],[1]) = \{000, 001, 011, 111\}.$$

Each of these has at least one repetition, which means that each of these simplices is degenerate.

Example 2.11. For Δ^2, we can picture the 0-, 1-, and 2-simplices in a similar way:



Proof. (of Definition and Lemma 2.9) We need to verify the simplicial identities. These follow from similar identities for the maps δ_k and σ_k, called *cosimplicial identities*. Specifically,

$$\begin{aligned}
\delta_j\delta_i &= \delta_i\delta_{j-1}, \text{ for all } i<j,\\
\sigma_j\sigma_i &= \sigma_{i-1}\sigma_j, \text{ for all } i>j,\\
\sigma_j\delta_i &= \begin{cases} \delta_i\sigma_{j-1} & i<j \\ \mathrm{id} & i=j \text{ or } i=j+1 \\ \delta_{i-1}\sigma_j & i>j+1 \end{cases}
\end{aligned} \tag{2.12}$$

These follow directly from the definitions: for example, we check $\delta_j\delta_i = \delta_i\delta_{j-1}$ for $i<j$. The image of δ_i (in this order) is $0,1,\dots,i-1,i+1,\dots,n$. Here $i+1$ is in the i-th spot. For $j>i$, the composite $\delta_j\delta_i$ therefore has the image (in this order) $0,1,\dots,i-1,i+1,\dots,j-1,j+1,\dots n$. On the other hand, δ_{j-1} has image $0,1,\dots,j-2,j,\dots,n$. Here $j-2$ is at the j-th spot, which comes after the i-th spot (for $i<j$). Thus, $\delta_i\delta_{j-1}$ has image

$0, \dots, i-1, i+1, \dots, j-2+1, j+1, \dots, n$, which is the same as the first map.

Next, σ_i repeats i, and $\sigma_j \sigma_i$ repeats (for $j < i$) j and i. On the other hand, σ_j repeats j and $\sigma_{i-1}\sigma_j$ also repeats i and j (note the shift because of $i - 1 \geqslant j$).

The remaining identities can be checked in a similar manner(!).

Now, these cosimplicial identites transform into the simplicial identities: for example, in order to show the simplicial identity

$$d_i d_j = d_{j-1} d_i$$

for $i < j$, we take $f \in (\Delta^e)_n = \mathrm{Hom}_\Delta([n], [e])$. By definition of the face maps, we have $d_i d_j(f) = d_i(f \circ \delta_j) = f \circ \delta_j \circ \delta_i$. (Note how the order of i and j changed.) By the cosimplicial identity, this equals $f \circ \delta_i \circ \delta_{j-1}$, which is $d_{j-1}(f \circ \delta_i) = d_{j-1} d_i(f)$. The same argument works for the other identities, always using that σ_k corresponds to s_k and δ_k to d_k, and that the order of composition is reversed when passing from the cosimplicial to the simplicial identities. □

2.2 From topological spaces to simplicial sets

Simplicial sets are important in algebraic topology because they mediate between topological spaces and (eventually) abelian groups. For a given topological space X, we want to define a simplicial set $\mathrm{Sing}(X)$ whose 0-simplices are the points of X, whose 1-simplices are continuous paths in X etc.

Definition 2.13. The (topological) *n-simplex* Δ^n is defined as

$$\Delta^n := \Delta^n_{\mathrm{Top}} := \{(t_0, \dots, t_n) \in \mathbf{R}^{n+1}, t_k \geqslant 0, \sum_k t_k = 1\}.$$

Remark 2.14. Note that Δ^0 is just a point, and $\Delta^1 = \{(t, 1-t), t \in [0,1]\}$ is homeomorphic to the closed unit interval $[0,1] \subset \mathbf{R}$. The above definition of Δ^1, however, is more symmetric. The condition that $x_k \geqslant 0$ can be dropped without ultimately changing anything in the results in this course. The condition is there mainly to simplify drawing pictures.

Remark 2.15. The symbol Δ has now been used in relation to three different entities:

- the simplex category Δ (Definition 2.8),
- the topological n-simplex $\Delta^n := \Delta^n_{\text{Top}} \in \text{Top}$ (Definition 2.13),
- and the simplicial set $\Delta^n := \Delta^n_{\text{simp}}$ (Definition and Lemma 2.9).

The simplex category plays on a completely different floor than the other two, but the two Δ^n are closely related, as we will see. We trust there is no confusion which meaning is intended.

We define (continuous) maps, called *face maps*

$$\delta_k : \Delta^n \to \Delta^{n+1}, \quad (k = 0, \dots, n+1) \tag{2.16}$$

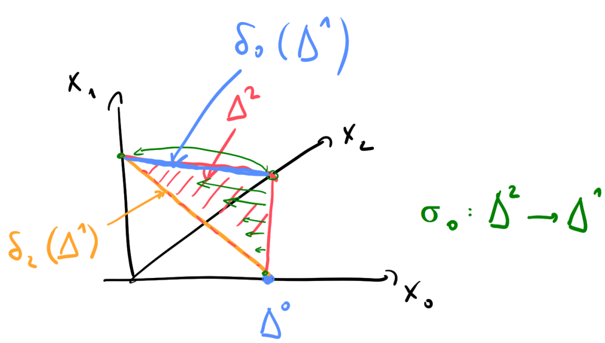
by $\delta_k((t_0, \dots, t_n)) := (t_0, \dots, t_{k-1}, 0, t_k, \dots, t_n)$, i.e., insert a 0 into the k-th spot. Thus, δ_k inserts Δ^n into Δ^{n+1} *opposite* to the k-th vertex. We define so-called *degeneracy maps*

$$\sigma_k : \Delta^{n+1} \to \Delta^n, 0 \leqslant k \leqslant n$$

by

$$\sigma_k(t_0, \dots, t_{n+1}) = (t_0, \dots, t_{k-1}, t_k + t_{k+1}, \dots, t_{n+1}).$$

Thus, σ_k contracts the k-th boundary.



Lemma 2.17. These maps satisfy the same relations as in (2.12), for example $\sigma_j \delta_i = \delta_i \sigma_{j-1}$ for $i < j$.

Proof. This is(!)a routine check. For example, we check the one highlighted above: $\delta_i(x_0, \dots, x_n) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_n)$ and $\sigma_j \delta_i(x_0, \dots, x_n) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{j-1}{+}x_j, x_{j+1}, \dots, x_n)$. On the other hand $\sigma_{j-1}(x_0, \dots, x_n) = (x_0, \dots, x_{j-1} + x_j, x_{j+1}, \dots)$, and δ_i inserts a zero in some spot at or before the $x_{j-1} + x_j$, so we get the same expression as before. $\square$

Example 2.18. Let X be a topological space. Define a simplicial set $\mathrm{Sing}(X)$, called the *singular simplicial set* or *singular simplicial complex* of X, by

$$\mathrm{Sing}_n(X) := \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$$

(continuous maps from the topological n-simplex, Definition and Lemma 2.9, to X). The face and degeneracy maps are all induced from similar maps for the Δ^n. E.g., for $f : \Delta^n \to X$,

$$d_k(f) \in \mathrm{Sing}_{n-1}(X)$$

is the map

$$\Delta^{n-1} \xrightarrow{\delta_k} \Delta^n \xrightarrow{f} X.$$

More concretely,

$$\begin{aligned}(d_k(f))(t_0, \ldots, t_{n-1}) &= f(t_0, \ldots, t_{k-1}, 0, t_k, \ldots),\\ (s_k(f))(t_0, \ldots, t_n) &= f(t_0, \ldots, t_{k-1}, t_k + t_{k+1}, \ldots, t_n).\end{aligned}$$

Our eventual goal is to extract crucial information from topological spaces using these simplicial sets. For the moment, note only that $\mathrm{Sing}_n(X)$ is a huge set, making it nearly impossible to do any computations with this directly: the simplicial set S^1_{simp} sketched in Example 2.7 is much smaller, and thus much more useful than $\mathrm{Sing}(S^1_{\mathrm{Top}})$ (here $S^1_{\mathrm{Top}} = \{z = x + iy \in \mathbf{C}, |z|^2 = x^2 + y^2 = 1\}$ is the circle, a topological space). We do have a map

$$S^1_{\mathrm{simp}} \to \mathrm{Sing}(S^1_{\mathrm{Top}})$$

mapping $*$ to $(1, 0)$ and γ to the loop

$$\Delta^1 \to S^1_{\mathrm{Top}}, (t, 1 - t) \mapsto \exp(2\pi i t).$$

A key insight is that, nonetheless, these two simplicial sets are not so different: we will eventually show that the homology of these two simplicial sets is the *same*.

The only case we can handle at this point is a point:

Example 2.19. $\mathrm{Sing}(\{*\}) = \Delta^0$. Indeed, any map $\Delta^n_{\mathrm{Top}} \to \{*\}$ is just constant. What can you say about $\mathrm{Sing}(X)$, where X is a *discrete* topological space (all subsets $U \subset X$ are open)?

Remark 2.20. A slightly more high-level formulation of the Sing-functor is to observe that the topological spaces Δ^n assemble into a *cosimplicial topological space*, i.e., a functor

$$\Delta \stackrel{F}{\to} \mathrm{Top}, [n] \mapsto \Delta^n.$$

The functor $\mathrm{Sing}(X)$ is then the composite

$$\Delta^{\mathrm{op}} \stackrel{F}{\to} \mathrm{Top}^{\mathrm{op}} \stackrel{\mathrm{Hom}_{\mathrm{Top}}(-,X)}{\to} \mathrm{Set}.$$

Similar cosimplicial objects appear in other mathematical domains. For example the cosimplicial object in schemes,

$$\Delta \to \mathrm{Sch}, [n] \mapsto \Delta^n_{\mathrm{alg}} := \mathrm{Spec}(\mathbf{Z}[t_0, \dots, t_n]/\sum t_i = 1).$$

plays a vital rôle in so-called $\mathbf{A}^1$-homotopy theory, a branch of algebraic geometry.

2.3 Simplicial sets as functors

Since Definition 2.1 is quite verbose, it is helpful to recast the definition using a functorial definition.

Definition and Lemma 2.21. Let C be a category. A *simplicial object* in C is the same as a functor

$$X : \Delta^{\mathrm{op}} \to C,$$

while a *semi-simplicial object* in C is the same as a functor

$$X : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to C.$$

Given such a functor, we often write

$$X_n := X([n]), \alpha^* := X(\alpha) : X_n \to X_m$$

for $\alpha : [m] \to [n]$. We also write $d_i = \delta_i^*$, $s_i = \sigma_i^*$.

Proof. Any functor $X : \Delta^{\mathrm{op}} \to C$ gives rise to a simplicial object as in Definition 2.1, because of the identities satisfied by the maps δ_k and σ_k in (2.12).

We sketch the converse, referring to [Mac98, §VII.5] for a detailed exposition. One proves (without much trouble) that any map $\alpha : [m] \to [n]$ in Δ factors uniquely as

$$\alpha = \underbrace{\delta_{i_1} \circ \cdots \circ \delta_{i_k}}_{=\alpha_{\text{inj}}} \circ \underbrace{\sigma_{j_1} \circ \sigma_{j_h}}_{=\alpha_{\text{surj}}} \tag{2.22}$$

with $m + k = n + h$ and $n > i_1 > \cdots > i_k \geqslant 0$ and $0 \leqslant j_1 < \cdots < j_h < m - 1$. In fact, $\alpha = \alpha_{\text{inj}} \circ \alpha_{\text{surj}}$ is the standard factorization of a map into a an injective after a surjective map. For example, for $i \leqslant j$ the map $\delta_i \delta_j$ can be put into this form, namely $\delta_{j+1}\delta_i$, by the cosimplicial identities (2.12). Thus, given a simplicial object X as defined in Definition 2.1, one defines $X(\alpha) : X_n \to X_m$ to be the composite

$$X_n \xrightarrow{d_{i_1}} X_{n-1} \cdots \xrightarrow{d_{i_k}} X_{n-k} \xrightarrow{s_{j_1}} X_{n-k-1} \cdots \xrightarrow{s_{j_h}} X_{n-k-h} = X_m.$$

Hereafter, we will only use the presentation of simplicial sets as in Definition and Lemma 2.21. As with other mathematical notions, it is useful to consider simplicial sets not just in isolation, but rather as objects in some category.

Definition 2.23. Given a category C, the category of *simplicial objects* in C is defined as

$$\mathrm{s}C := \mathrm{Fun}(\Delta^{\mathrm{op}}, C),$$

the functor category of functors from Δ^{op} to C. Thus an object in sC is just a simplicial object as defined before. A morphism $f : X \to Y$ of simplicial objects is a collection of maps $f_n : X_n \to Y_n$ such that for each $\alpha : [m] \to [n]$, the diagram

$$\begin{tikzcd} X_n \arrow[r, "\alpha^*"] \arrow[d, "f_n"] & X_m \arrow[d, "f_m"] \\ Y_n \arrow[r, "\alpha^*"] & Y_m \end{tikzcd}$$

commutes.

We will apply this in particular to $C = \mathrm{Set}$ and $C = \mathrm{Ab}$, which gives us the categories sSet of *simplicial sets* and sAb of *simplicial abelian groups*.

Dually, a *cosimplicial object* is a functor $\Delta \to C$.

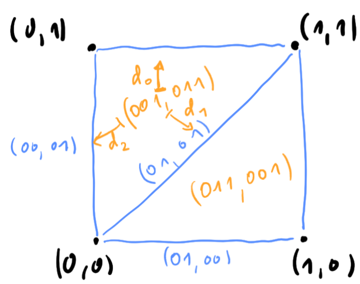
By Definition and Lemma 2.21, specifically (2.22), it suffices to check this commutativity condition for α being either some δ_k or some σ_k.

Definition 2.24. Given two simplicial sets X, Y, the *product* $X \times Y$ is the simplicial set defined as

$$(X \times Y)_n = X_n \times Y_n,$$

with $\alpha^*_{X \times Y}$ being the product of α^*_X and α^*_Y.

Example 2.25. The simplicial set $\Delta^1 \times \Delta^1$ looks as follows :



More formally, we list the simplices, where we use a notation $(\dots,\dots)$ to denote a pair of simplices in Δ^1:

$$\begin{aligned}
(\Delta^1 \times \Delta^1)_0 =& \{(0,0),(0,1),(1,0),(1,1)\} \\
(\Delta^1 \times \Delta^1)_1 =& \{(00,00)^*,(01,00),(11,00)^*,(11,01),(11,11)^*,(00,01),(00,11)^*,(01,11)\}, \\
(\Delta^1 \times \Delta^1)_2 \ni& (011,001),(001,011),
\end{aligned}$$

(1-simplices with a $*$ are degenerate; there are many more degenerate 2-simplices; note that $(011,001)$ is non-degenerate, even though both components are individually degenerate 2-simplices in Δ^1. See also Exercise 2.8.) In particular, $\Delta^1 \times \Delta^1$ is not isomorphic to Δ^2: the latter has exactly one non-degenerate 2-simplex, see above.

Definition 2.26. Given two simplicial sets X, Y, the *coproduct* $X \sqcup Y$ is the simplicial set defined as

$$(X \sqcup Y)_n = X_n \sqcup Y_n,$$

with $\alpha^*_{X \sqcup Y}$ being the coproduct (i.e., disjoint union) of the maps α^*_X and α^*_Y.

We will use these definitions in slightly greater generality, where instead of two simplicial sets, we allow a family $(X_i)_{i\in I}$ of simplicial sets. The above definitions carry over verbatim.

Definition 2.27. Given three simplicial sets X, Y, Z and simplicial maps f, g as depicted

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \\ Z, & \end{tikzcd}$$

the *pushout* $Y \sqcup_X Z$ is the simplicial set with

$$(Y \sqcup_X Z)_n := Y_n \sqcup_{X_n} Z_n.$$

I.e., the n-simplices of $Y \sqcup_X Z$ are the n-simplices $y \in Y_n$, $z \in Z_n$, where two such simplices are identified if there is an n-simplex in X, $x \in X_n$ that maps to y and z, respectively. Again, the maps $\alpha^*_{Y \sqcup_X Z}$ are induced from the ones on X, Y, and Z.

Example 2.28. We can now complete the definition of the simplicial circle begun in Example 2.7. We define S^1 to be the pushout of the diagram

$$\begin{tikzcd} \Delta^0 \sqcup \Delta^0 \arrow[r, "i"] \arrow[d, "p"] & \Delta^1 \\ \Delta^0 & \end{tikzcd}$$

Here the map p is the obvious projection map and i is the inclusion of the two endpoints of Δ^1: more formally,

$$(\Delta^0 \sqcup \Delta^0)_n = \{*\} \sqcup \{*\} \to \Delta^1_n$$

sends the first point to the map $[n] \to [1]$ mapping everything to 0, and to 1, respectively. In degrees 0 and 1, we thus have

$$S^1_0 = \{*\} \sqcup_{\{*\}\sqcup\{*\}} \{0, 1\} = \{*\},$$

$$S^1_1 = \{*\} \sqcup_{\{*\}\sqcup\{*\}} \{00, 01, 11\},$$

which means we identify 00 and 11. This reproduces Example 2.7.

Remark 2.29. The general paradigm at work in the above definition is the following: suppose C is a category that has all (small) limits or colimits, such as $C = \mathrm{Set}$. Then, for any small category D, the functor category $\mathrm{Fun}(D, C)$ has all limits or colimits, and the evaluation functors

$$\mathrm{Fun}(D, C) \stackrel{\mathrm{ev}_d}{\to} C, f \mapsto f(d)$$

preserve these limits or colimits. Applied to $D = \Delta^{\mathrm{op}}$ and $C = \mathrm{Set}$, this gives the above notion of products and coproducts.

Lemma 2.30. The functor $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$ preserves products and coproducts.

That is, for a family $(X_i) \in \mathrm{Top}$, the following natural maps (of simplicial sets) are isomorphisms:

$$\mathrm{Sing}(\prod_i X_i) \stackrel{\cong}{\to} \prod_i \mathrm{Sing}(X_i),$$
$$\bigsqcup_i \mathrm{Sing}(X_i) \stackrel{\cong}{\to} \mathrm{Sing}(\bigsqcup X_i).$$

Remark 2.31.

- These maps are given on n-simplices as follows: an n-simplex in $\prod X_i$ is a continuous map $\Delta^n \to \prod X_i$. For each $j \in I$, the composite with the (continuous) projection map $\prod X_i \to X_j$ gives an element in $\mathrm{Sing}_n(X_j)$, which in total is an n-simplex in the right hand side. A similar description holds (!)for the second map, using the (continuous) injections $X_j \to \bigsqcup_i X_i$ instead.
- We will use the product part later in the proof of the homotopy axiom, see Proposition 2.39, while the coproduct part will be instrumental in proving the additivity axiom (Proposition 4.4).

Proof. We use that for any topological space T,

$$\mathrm{Hom}_{\mathrm{Top}}(T, \prod_i X_i) = \prod_i \mathrm{Hom}_{\mathrm{Top}}(T, X_i).$$

Indeed, a map $f : T \to \prod_i X_i$ is tantamount to a family of maps $f_i : T \to X_i$. By the characterization of the product topology [Hat, §1] f is continuous iff all the f_i are continuous. We apply this to $T = \Delta^n$ and get the required bijection.

As for the second map, we immediately see that it is injective. Let $f : \Delta^n \to \bigsqcup X_i =: X$ be a continuous map. We need to show there is some $j \in I$ such that $f(\Delta^n)$ is contained in $X_j \subset X$. Otherwise, there is j and $j' \in I$ such that $f^{-1}(X_j)$ and $f^{-1}(X_{j'})$ are both nonempty. Then $X = X_j \sqcup \left(\bigsqcup_{i \in I, i \neq j} X_i\right) =: X_j \sqcup X'$ is a disjoint union of two nonempty open subsets (by definition of the topology on the disjoint union). Then

$$\Delta^n = f^{-1}(X_j) \sqcup f^{-1}(X')$$

is a disjoint union of two open subsets. This contradicts the fact that Δ^n is path-connected and therefore, see e.g. [Hat, §2], connected. □

Remark 2.32. Note the first part of the proof is very formal. It uses only that $\mathrm{Hom}_{\mathrm{Top}}(\Delta^n, -)$ turns products into products. (More generally, it is true that it preserves limits. Thus, Sing preserves limits.) By contrast, the second statement has a peculiar proof, which also does not extend much further: for a topological space $X = U \cup V$ (for two subspaces U and V), we have

$$\mathrm{Sing}(X) \neq \mathrm{Sing}(U) \cup \mathrm{Sing}(V)!$$

Indeed, an n-simplex in $\mathrm{Sing}(X)$ need not lie in either $\mathrm{Sing}(U)$ or $\mathrm{Sing}(V)$: a map $\Delta^n_{\mathrm{Top}} \to X$ need not factor over U or V. The bulk of our later work on the excision property will be to salvage this problem.

In addition to understanding simplicial sets properly, we also need to understand maps between them properly. To this end, we use a general lemma from category theory, called the *Yoneda lemma*, see Lemma A.1. Specialized to our situation, it says the following:

Lemma 2.33. Let X be a simplicial set. Then there is a bijection

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \to X_n (:= X([n])$$

which takes a map $f : \Delta^n \to X$, takes its evaluation at n, $f_n : (\Delta^n)_n = \mathrm{Hom}_\Delta([n],[n]) \to X_n$ and takes the image of the identity map $\mathrm{id}_{[n]}$, which gives an element in X_n.

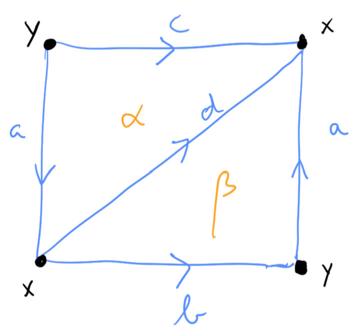
Proof. This is just Lemma A.1, applied to $C = \Delta$ (so that $\mathrm{sSet} = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$, and using that, by definition, Δ^n is the representing functor associated to $[n] \in \Delta$. □

Corollary 2.34. There is a bijection

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \Delta^m) \to \mathrm{Hom}_\Delta([n],[m]).$$

Both in the form of Lemma 2.33 and Corollary 2.34, the Yoneda lemma is a highly useful device to construct maps between simplicial sets. We illustrate this by constructing the (simplicial) *Möbius strip*.

Example 2.35. Our goal is to construct a simplicial set M such that $M_0 = \{x, y\}$, while M_1 contains 4 non-degenerate edges a, b, c, d, and M_2 contains 2 non-degenerate simplices α and β such that the face maps d_k have the behaviour as depicted:



It is of course possible to "manually" specify the n-simplices of M for all n, and define face and degeneracy maps by hand etc. However, this is tedious and geometrically unenligthening. We will therefore instead construct M in two steps; both steps will be a pushout of simplicial sets we already know.

- We begin the construction by glueing two copies of Δ^2 along an edge. To this end, consider the following diagram of simplicial sets:

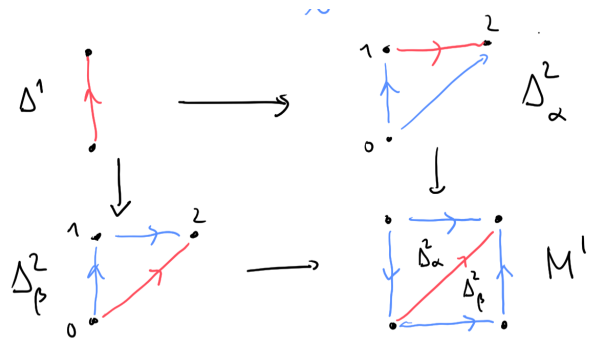
$$\begin{tikzcd} \Delta^1 \arrow[r, "12"] \arrow[d, "02"] & \Delta^2_\alpha \arrow[d, "i_\alpha"] \\ \Delta^2_\beta \arrow[r, "i_\beta"] & M' \end{tikzcd}$$

  Here, the subscripts at the Δ^2's just serve as a label. The maps $ij : \Delta^1 \to \Delta^2$ are the maps that correspond to the element $ij \in \mathrm{Hom}_\Delta([1], [2]) (= \{ij, 0 \leqslant i \leqslant j \leqslant 2\}$. We define a simplicial set M' to be the pushout of this diagram, i.e.,

$$M' := \Delta^2 \sqcup_{12, \Delta^1, 02} \Delta^2,$$

  where the subscripts in the $\sqcup$ indicate that the pushout is formed along these maps $\Delta^1 \to \Delta^2$. The maps i_α and i_β are the canonical maps into the pushout (the labels just serve to remember which copy of Δ^2 is which).

The simplicial set M' can be pictured as follows:

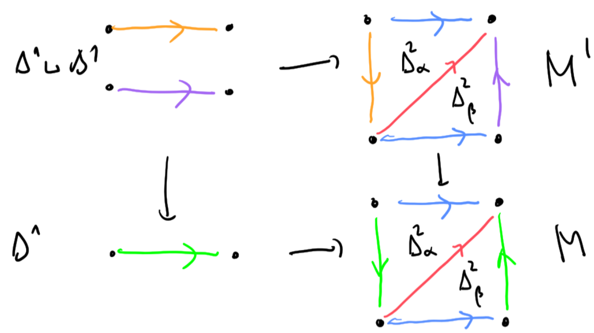


- By construction, M' has 4 vertices, and 5 non-degenerate edges. We intend to further identify 2 edges (and, therefore, certain vertices). To this end, define M to be the pushout of the diagram

$$\begin{tikzcd}
\Delta^1 \sqcup \Delta^1 \arrow[r, "((01)_\alpha,(12)_\beta)"] \arrow[d, "\mathrm{id}\sqcup\mathrm{id}"] & M' \arrow[d] \\
\Delta^1 \arrow[r] & M,
\end{tikzcd}$$

or, in more compact notation

$$M = M' \sqcup_{(01,1'2'),\Delta^1\sqcup\Delta^1,\mathrm{id}\sqcup\mathrm{id}} \Delta^1.$$



The horizontal map above is composed of two maps $\Delta^1 \to M'$. We define these maps as

$$\Delta^1 \xrightarrow{01} \Delta^2 \xrightarrow{i_\alpha} M',$$

$$\Delta^1 \xrightarrow{12} \Delta^2 \xrightarrow{i_\beta} M'.$$

In other words, we identify the edge 01 in the "α"-copy of Δ^2 with the edge 12 in the "β"-copy of Δ^2.

Alternatively, to specify these maps, it is possible to use the Yoneda lemma once again:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{sSet}}(\Delta^1, M') &\cong M'_1 \\ &= (\Delta^2_\alpha)_1 \sqcup_{12,(\Delta^1)_1,02} (\Delta^2_\beta)_1.\end{aligned}$$

2.4 Continuous and simplicial homotopies

Recall the product of simplicial sets from Definition 2.24. We will use Corollary 2.34, so that the maps $\delta_k : [0] \to [1]$ ($k = 0, 1$) give rise to a map $\Delta^0 \to \Delta^1$, again denoted by δ_k.

Definition 2.36. Let $f, g : X \to Y$ be two maps between simplicial sets. A *simplicial homotopy* between f and g is a map

$$h : \Delta^1 \times X \to Y$$

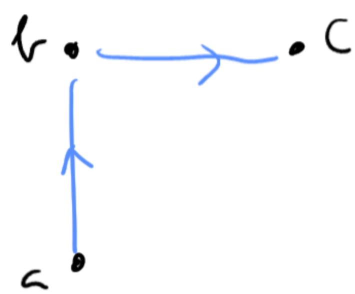
such that the following diagram commutes:

$$\begin{tikzcd}
\Delta^0 \times X \arrow[r, equal] \arrow[d, "\delta^0"'] & X \arrow[dr, "f"] & \\
\Delta^1 \times X \arrow[rr, "h"] & & Y \\
\Delta^0 \times X \arrow[r, equal] \arrow[u, "\delta^1"] & X \arrow[ur, "g"'] &
\end{tikzcd} \quad . \tag{2.37}$$

A simplicial map $f : X \to Y$ is called a *simplicial homotopy equivalence*, if there is a map $g : Y \to X$ and simplicial homotopies between id_X and $g \circ f$ as well as between id_Y and $f \circ g$.

Remark 2.38.
- Definition 2.36 looks conspicuously similar to the standard definition of continuous homotopies between continuous maps. Indeed, given two continuous maps $f, g : X \to Y$ between two topological spaces, a homotopy is a continuous map h as above, but where now Δ^1 is the standard 1-simplex. (In many textbooks, homotopies are defined as maps $[0, 1] \times X \to Y$, but $[0, 1]$ is homeomorphic to Δ^1, so there is no difference.)

- A difference between homotopies in sSet and Top is that the relation "f is homotopic to g" fails to be transitive (and is therefore not an equivalence relation) for simplicial homotopies. For example, let $Y = \Delta^1 \sqcup_{\Delta^0} \Delta^1$ and $X = \Delta^0$. There are three 0-simplices (equivalently, maps $\Delta^0 \to Y$) a, b and c as pictured, and a is homotopic to b, and b homotopic to c. Yet, there is no 1-simplex whose boundaries would be a and c.



(On the positive side, if Y is a so-called *Kan complex*, then the homotopy relation *is* an equivalence relation, see [GJ09, Corollary I-6.2]. The Y above fails that additional condition.)

In contrast, the glueing lemma in topology quickly implies that homotopy is an equivalence relation [Rot88, Theorem 1.2].

Proposition 2.39. The Sing-functor preserves homotopies. More formally: any continuous homotopy $h : X \times \Delta^1 \to Y$ between two continuous maps $f, g : X \to Y$ gives rise to a (simplicial) homotopy between

$$\mathrm{Sing}(f) \text{ and } \mathrm{Sing}(g) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y).$$

Proof. Suppose we have a diagram as in (2.37), where all objects are topological spaces and all maps continuous. The functor Sing preserves products (Lemma 2.30), so that

$$\mathrm{Sing}(\Delta^1_{\mathrm{Top}} \times X) \cong \mathrm{Sing}(\Delta^1_{\mathrm{Top}}) \times \mathrm{Sing}(X).$$

Here we write Δ^1_{Top} for the topological 1-simplex. The identity $\Delta^1_{\mathrm{Top}} \to \Delta^1_{\mathrm{Top}}$ is a 1-simplex in $\mathrm{Sing}(\Delta^1_{\mathrm{Top}})$, or equivalently, by the Yoneda lemma (Lemma 2.33), a map of simplicial sets

$$\Delta^1 \to \mathrm{Sing}(\Delta^1_{\mathrm{Top}}).$$

Thus, applying Sing to (2.37), and composing with this map gives a diagram

$$\begin{tikzcd}
\Delta^0 \times \mathrm{Sing}(X) \arrow[r, equal] \arrow[d, "\delta^0"] & \mathrm{Sing} X \arrow[d, "\delta^0"] \arrow[dr, "\mathrm{Sing} f"] & \\
\Delta^1 \times \mathrm{Sing}(X) \arrow[r] & \mathrm{Sing}(\Delta^1_{\mathrm{Top}} \times X) \arrow[r, "\mathrm{Sing} h"] & \mathrm{Sing} Y \\
\Delta^0 \times X \arrow[r, equal] \arrow[u, "\delta^1"'] & \mathrm{Sing} X. \arrow[u, "\delta^1"'] \arrow[ur, "\mathrm{Sing} g"'] &
\end{tikzcd}$$

Thus, $\mathrm{Sing} f$ is (simplicially) homotopic to $\mathrm{Sing} g$. $\square$

2.5 From simplicial sets to topological spaces: the geometric realization

In this section, we are going to construct a functor

$$| - | : \mathrm{sSet} \to \mathrm{Top},$$

which gives a precise meaning to the idea that to each simplicial set corresponds some "picture," i.e., a topological space.

Definition 2.40. For a set T, we regard T as a topological space with the discrete topology, and $T \times \Delta^n_{\mathrm{Top}}$ (for some n) carries the product topology. The *geometric realization* $|X|$ is the following topological space:

$$\bigsqcup_{n \geqslant 0} X_n \times \Delta^n_{\mathrm{Top}} / \sim,$$

where $\sim$ is the equivalence relation generated by the following relation: a pair $(x_n, (t_0, \dots, t_n)) \in X_n \times \Delta^n_{\mathrm{Top}}$ is identified with a pair $(y_m, (u_0, \dots, u_m))$ if there is a map $\alpha : [m] \to [n]$ such that

$$\alpha^*(x_n) = x_m$$

and

$$\alpha_*((u_0, \dots, u_m)) = (t_0, \dots, t_n).$$

Recall that $\alpha^* : X_n \to X_m$ is the map given by evaluating $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ at α. Similarly, $\alpha_* : \Delta^m_{\mathrm{Top}} \to \Delta^n_{\mathrm{Top}}$ is given by evaluating the functor $\Delta \to \mathrm{Top}$ mentioned in Remark 2.20. We equip $|X|$ with the quotient topology (i.e., a subset $U \subset |X|$ is open iff its preimage in all the $X_n \times \Delta^n_{\mathrm{Top}}$ is open. Equivalently, a map $|X| \to T$ to any other topological space T is continuous iff its restriction to all the $X_n \times \Delta^n_{\mathrm{Top}}$ is continuous.)

Remark 2.41.
- Since every morphism α is the composite of maps δ_k and σ_k (cf. the proof of Definition and Lemma 2.21), it is enough to consider the relation as above in which either $\alpha = \delta_k : [n-1] \to [n]$ or $\alpha = \sigma_k : [n] \to [n-1]$ (for appropriate n, k). Thus, writing $\underline{u} = (u_0, \dots, u_n) \in \Delta^n_{\mathrm{Top}}$,

$$X_n \times \Delta^n_{\mathrm{Top}} \ni (x_n, \underbrace{(u_0, \dots, u_{k-1}, 0, u_k, \dots u_n)}_{=\delta_k(\underline{u})}) \sim (d_k(x_n), \underline{u}) \in X_{n-1} \times \Delta^{n-1}_{\mathrm{Top}},$$

$$X_{n-1} \times \Delta^n_{\mathrm{Top}} \ni (x_{n-1}, \underbrace{(u_0, \dots, u_{k-1}, u_k + u_{k+1}, \dots, u_n)}_{=\sigma_k(\underline{u})}) \sim (s_k(x_{n-1}), \underline{u}) \in X_n \times \Delta^n_{\mathrm{Top}}.$$

- In particular, for a degenerate simplex $s_k(x_n) \in X_{n+1}$, the subspace $\{s_k(x_n)\} \times \Delta^{n+1}_{\mathrm{Top}}$ is identified with $\{x_n\} \times \Delta^n$, since $\sigma_k : \Delta^{n+1}_{\mathrm{Top}} \to \Delta^n_{\mathrm{Top}}$ is surjective.

Remark 2.42. The geometric realization functor has the following properties. For proofs, one can consult [FP90, §4.3].

- For $X = \Delta^1_{\mathrm{simp}}$, we have a homeomorphism $|\Delta^1| \cong \Delta^1_{\mathrm{Top}}$. Indeed, the subspace $\{ii\} \times \Delta^1_{\mathrm{Top}}$ ($i = 0, 1$) corresponding to the two degenerate 1-simplices is identified with $\{i\} \times \Delta^0_{\mathrm{Top}}$. On the other hand, $0 = d_1(01)$ so that $\{0\} \times \Delta^0_{\mathrm{Top}}$ is identified with $\{01\} \times (1, 0) \in \{01\} \times \Delta^1_{\mathrm{Top}}$ and similarly $\{1\} \times \Delta^0_{\mathrm{Top}} \sim \{01\} \times (0, 1)$.
- More generally, there is a homeomorphism

$$|\Delta^n| \cong \Delta^n_{\mathrm{Top}}.$$

- For a coproduct of simplicial sets X_i, $|\bigsqcup X_i| = \bigsqcup |X_i|$.
- For any map of simplicial sets $f : X \to Y$, we have a continuous map $|f| : |X| \to |Y|$ that sends (the equivalence class of) $(x_n, \underline{t})$ to $(f(x_n), \underline{t})$. This is well-defined since f is functorial, i.e., $d_k(f(x)) = f(d_k(x))$ and likewise with s_k.

- For a pushout, we have a natural homeomorphism

$$|Y| \sqcup_{|X|} |Z| \xrightarrow{\cong} |Y \sqcup_X Z|.$$

  For example, for the simplicial n-sphere (cf. Exercise 2.7)

$$S^n_{\text{simp}} := \partial\Delta^n \sqcup_{\Delta^0} \Delta^n,$$

  we have

$$|S^n_{\text{simp}}| = \partial\Delta^n_{\text{Top}} \sqcup_{\Delta^0_{\text{Top}}} \Delta^n_{\text{Top}},$$

  so that there is a homeomorphism to the (topological) n-sphere:

$$|S^n_{\text{simp}}| \cong S^n_{\text{Top}}.$$

  Another example: the geometric realization of the (simplicial) Möbius strip (Example 2.35) is obtained by glueing two copies of Δ^2_{Top} along an edge (exactly the same way as above), and then by identifying two edges with another. Thus, the geometric realization is homeomorphic to the usual Möbius strip:

$$|M| \cong [0,1] \times [0,1]/(t,0) \sim (1-t,1).$$

- By the last two properties, $X \mapsto |X|$ is a colimit-preserving functor. By general category theory there is, up to a unique isomorphism, only one colimit-preserving functor sSet $\to$ Top with the property that its restriction to the full subcategory $\Delta \subset \text{sSet} = \text{Fun}(\Delta^{\text{op}}, \text{Set})$ (via the Yoneda embedding) agrees with the functor $[n] \mapsto \Delta^n_{\text{Top}}$:

$$\begin{tikzcd}
\Delta \arrow[r, "{[n]\mapsto\Delta^n_{\text{Top}}}"] \arrow[d, hook, "\text{Yoneda}"'] & \text{Top} \\
\text{sSet.} \arrow[ur, dotted, "|-|"'] &
\end{tikzcd}$$

Outlook 2.43. The two functors

$$|-| : \text{sSet} \rightleftarrows \text{Top} : \text{Sing}. \tag{2.44}$$

are adjoint functors, i.e.,

$$\text{Hom}_{\text{Top}}(|X|, Y) = \text{Hom}_{\text{sSet}}(X, \text{Sing}(Y))$$

functorial in $X \in \mathrm{sSet}$ and $Y \in \mathrm{Top}$. This can be shown by reducing the claim to $X = \Delta^n$, where it holds by

$$\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, Y) = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{Top}}, Y) =: \mathrm{Sing}(Y)_n \overset{Lemma\ 2.33}{=} \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, Y). \blacksquare$$

These two functors are *very far* from being an equivalence. Indeed,

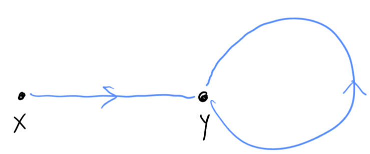
$$\mathrm{Sing}(|\Delta^1|) = \mathrm{Sing}(\Delta^1_{\mathrm{Top}})$$

is not at all isomorphic to Δ^1 (already for cardinality reasons). Conversely, it can be shown that $|X|$ is always a CW-complex, and not every space is homeomorphic to a CW-complex. A foundational result in homotopy theory states, however, that the adjunction (2.44) becomes an equivalence after inverting maps that induce isomorphisms on all homotopy groups π_n. See [GJ09, Theorem I.11.4] for the precise statement and proof.

2.6 Exercises

Exercise 2.1. Spell out the relation between the face and degeneracy maps between (topological) simplices. Use this to verify the remaining condition in Definition 2.1 in order to verify that $\mathrm{Sing}(X)$ is indeed a simplicial set.

Exercise 2.2. Using appropriate pushouts, define a simplicial set which corresponds to the following picture. (Hint: do the construction step by step.)



Exercise 2.3. Let C be a category and $c \in C$ an object. The *discrete simplicial* object, momentarily denoted by $\tilde{c}$ (but later just denoted by c) such that $(\tilde{c})_n = c$, and face and degeneracy maps are just the identity.

- What is the geometric realization of a discrete simplicial set?
- Redefine this using the functorial language (you will not need to use the words face or degeneracy maps).

Exercise 2.4. This exercise is the first ϵ towards the definition of ∞-categories. Let C be a small category (i.e., it has a set of objects, as opposed to a class). The *nerve* of C is the simplicial set

$$N(C) : \Delta^{\mathrm{op}} \to \mathrm{Set}, [n] \mapsto \mathrm{Hom}_{\mathrm{Cat}}([n], C),$$

where $[n]$ is regarded as a category in the natural way (objects given by k, $0 \leqslant k \leqslant n$ and there is precisely one morphism from k to l if $k \leqslant l$ and no morphism otherwise). Thus vertices of $N(C)$ are the objects, edges are morphisms.

- Consider the ordered set $[n]$ with its usual ordering, and thus as a category (in which $\mathrm{Hom}_{[n]}(i, j) = \{*\}$ if $i \leqslant j$ and the Hom-set is empty otherwise). Show that
$$\Delta^n = N([n]).$$
- Turn the following statement into a precise assertion "2-simplices of $N(C)$ are pairs of composable morphisms."
- Show that $C \mapsto N(C)$ is a functor $\mathrm{Cat} \to \mathrm{sSet}$.
- Show this functor is fully faithful.
- (Optional) Describe the essential image of N.

Exercise 2.5. Prove Corollary 2.34 yourself for $n = 0$, $m = 1$ by directly inspecting the two sets in question.

Exercise 2.6. Describe the non-degenerate simplices of the *simplicial cylinder* $S^1 \times \Delta^1$, including a description of the face maps. Draw these simplices!

Exercise 2.7. We define the *boundary* of an n-simplex, $\partial\Delta^n$ to be the sub-simplicial set of Δ^n such that

$$(\partial\Delta^n)_m = \{f : [m] \to [n], \text{order-preserving}, \mathrm{im}\, f \subsetneq [n]\}.$$

Verify that this is indeed a simplicial set. ($\partial\Delta^2$ has three vertices and three non-degenerate edges). Define the *simplicial n-sphere* to be the pushout

$$S^n := \Delta^n \sqcup_{\partial\Delta^n} \Delta^0,$$

i.e., the boundary of Δ^n is contracted to a point. Show that S^n has precisely two non-degenerate simplices, one in dimension 0 and one in dimension n.

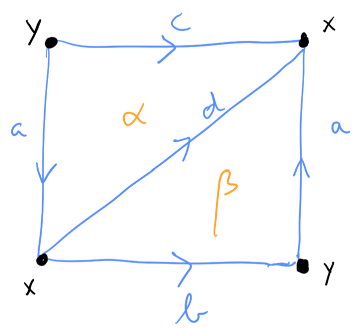
Exercise 2.8. Let $X = \Delta^1 \times \Delta^1$.

(1) How many 2-simplices does X have? Show that, as stated in Example 2.25, all except two of them are degenerate.

(2) Show (by a combinatorial consideration) that all 3-simplices of X are degenerate.

(3) Deduce (from (2) and the simplicial identities) that all k-simplices of X are degenerate for $k \geqslant 4$.

Exercise 2.9. Consider the (simplicial) circle S^1 and the (simplicial) Möbius strip M, which is the following simplicial set (cf. Example 2.35 for a formal construction):



- Show that there is precisely one map

$$i : S^1 \to M$$

such that the unique non-degenerate 1-simplex in S^1 gets mapped to a non-degenerate 1-simplex.

- Show that this map is a simplicial homotopy equivalence.

Chapter 3

Chain complexes

In this section, we introduce chain complexes and collect the pertinent basic insights from homological algebra. In depth-reference for this material include [Wei94; GM03]. In the overall architecture of (co)homology in algebraic topology, we consider a sequence of functors

$$\mathrm{Top} \stackrel{\mathrm{Sing}}{\to} \mathrm{sSet} \stackrel{\mathbf{Z}[-]}{\to} \mathrm{sAb} \stackrel{N}{\to} \mathrm{Ch} \stackrel{\mathrm{H}_n}{\to} \mathrm{Ab}.$$

In this chapter, we will introduce the category Ch of chain complexes, the normalized chain complex functor N, as well as the homology functors H_n.

3.1 Definitions

Definition 3.1. A *chain complex* is a sequence C_n $(n \in \mathbf{Z})$ of abelian groups, together with maps (called *differentials*

$$\partial_n : C_n \to C_{n-1}$$

such that the composition vanishes:

$$\partial_{n-1} \circ \partial_n = 0.$$

This condition is also referred to by saying that $\partial^2 = 0$. It is customary to drop ∂ from the notation and just say that C is a chain complex, leaving ∂ implicit.

A *chain map* between two chain complexes (C, ∂^C) and (D, ∂^D) is a sequence of homomorphisms of abelian groups $f_n : C_n \to D_n$

such that the following diagram commutes for all n:

$$\begin{tikzcd} C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ D_n \arrow[r, "\partial_n^D"] & D_{n-1}. \end{tikzcd} \tag{3.2}$$

Together with the obvious identity maps and the obvious composition, these form a category denoted by Ch.

For any ring R, chain complexes of R-modules are defined similarly with C_n being R-modules, and ∂_n and f_n being R-module maps. In the same vein, and yet more generally, there is a notion of chain complexes taking values in an abelian category A. These categories are denoted by $\mathrm{Ch}(\mathrm{Mod}_R)$ and $\mathrm{Ch}(A)$, respectively.

Definition 3.3. A *cochain complex* is a sequence (C^n), $n \in \mathbf{Z}$ of abelian groups and with differentials

$$\partial^n : C^n \to C^{n+1},$$

such that again $\partial^2 = 0$, i.e., $\partial^{n+1} \circ \partial^n = 0$ for all n.

Thus, the only difference to a chain complex is that the differentials have degree $+1$. Any chain complex (C_n) gives rise to a cochain complex defined by

$$C^n := C_{-n}, \partial^n := \partial_{-n},$$

i.e., just relabeling the components.

Example 3.4.

- The sequence

$$\ldots \to \mathbf{Z}/4 \xrightarrow{2} \mathbf{Z}/4 \xrightarrow{2} \mathbf{Z}/4 \ldots$$

(multiplication by 2 in each degree) is a chain complex, while the sequence

$$\ldots \to \mathbf{Z} \xrightarrow{\mathrm{id}} \mathbf{Z} \xrightarrow{\mathrm{id}} \mathbf{Z} \ldots$$

is *not* a chain complex, since $\mathrm{id} \circ \mathrm{id} \neq 0$.

- We can regard any abelian group M as a chain complex which is M in degree 0, and 0 otherwise (and all differentials are necessarily 0). We refer to this by saying that the complex is *concentrated* in degree 0.

- Given a chain complex C, the *shift* of C is defined as
$$C[p]_n := C_{n+p},$$
with differential $\partial_n^{C[p]} := (-1)^p \partial_{n+p}^C$. The relevance of the sign will become clear later in relation with the tensor product of chain complexes, Example 3.31.

- The name "differential" comes from analysis, where one shows that the process of taking the *exterior derivative* yields a cochain complex on, say, an open subset $M \subset \mathbf{R}^n$ (more generally, a differentiable manifold):
$$\Omega^*(M) : \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \dots . \tag{3.5}$$
Here $\Omega^0(M)$ denotes the vector space of smooth functions $M \to \mathbf{R}$ and $\Omega^k(M)$ denotes the vector space of (smooth) k-forms. This cochain complex is called the *de Rham complex*. The fact that $d^2 = 0$ ultimately relies on the fact that for a (twice differentiable) function $f : M \to \mathbf{R}$
$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$
A similar point arises in establishing the singular chain complexes, cf. the use of the simplicial identities in Definition and Lemma 3.8. This cochain complex is very closely related to the singular simplicial set $\mathrm{Sing}(M)$ (and the chain complexes that we will construct out of it in the next section). In fact, *Stokes' theorem* asserts that for an n-form $\omega \in \Omega^n(M)$ and an $n-1$-simplex $\sigma : \Delta^{n+1}_{\mathrm{Top}} \to M$, there holds
$$\int_{\delta\sigma} \omega = \int_\sigma d\omega.$$

3.2 From simplicial sets to chain complexes

In this section, we describe two functors
$$\mathrm{sSet} \xrightarrow{\mathbf{Z}[-]} \mathrm{sAb} \xrightarrow{N} \mathrm{Ch}$$
These are necessary ingredients to define singular homology of topological spaces.

We use the *free abelian group* functor

$$\begin{aligned}\mathbf{Z}[-] : \mathrm{Set} \to & \mathrm{Ab},\\ S \mapsto & \mathbf{Z}[S] = \{n : S \to \mathbf{Z}, n(s) = 0 \text{ for all but finitely many } s \in S\}.\end{aligned}$$

We denote such an element in $\mathbf{Z}[S]$ as the finite formal linear combination $\sum n_s s$ with $n_s (= n(s)) \in \mathbf{Z}$. For a map of sets $f : S \to T$, the induced map $\mathbf{Z}[S] \to \mathbf{Z}[T]$ is given by $\sum_{s \in S} n_s s \mapsto \sum_{s \in S} n_s f(s)$. Thus, it is the unique $\mathbf{Z}$-linear map sending $1 \cdot s$ to $1 \cdot f(s)$.

Remark 3.6. The functor $\mathbf{Z}[-]$ is left adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$, i.e., there is a natural bijection (for any set S and any abelian group A)

$$\mathrm{Hom}_{\mathrm{Ab}}(\mathbf{Z}[S], A) = \mathrm{Hom}_{\mathrm{Set}}(S, A).$$

In this bijection, a map $f : S \to A$ corresponds to the group homomorphism $g : \mathbf{Z}[S] \to A$ satisfying $g(\sum_{s \in S} n_s s) = \sum_{s \in S} n_s f(s)$. Conversely, a group homomorphism $g : \mathbf{Z}[S] \to A$ corresponds to the map (of sets) $S \stackrel{s \mapsto 1 \cdot s}{\to} \mathbf{Z}[S] \stackrel{g}{\to} A$.

Recall that for a category C, $\mathrm{s}C := \mathrm{Fun}(\Delta^{\mathrm{op}}, C)$ denotes the category of simplicial objects in C. The formation $C \mapsto \mathrm{s}C$ is functorial in C, in the following sense: given a functor $F : C \to D$, we get a functor

$$F : \mathrm{s}C \to \mathrm{s}D,$$

given by postcomposing with F. In particular, we get a functor

$$\mathbf{Z}[-] : \mathrm{sSet} \to \mathrm{sAb},$$

which concretely sends a simplicial set (X_n) to a simplicial abelian group whose n-simplices are $(\mathbf{Z}[X_n])$ and whose simplicial maps are given by functoriality of $\mathbf{Z}[-]$. That is, for $\alpha : [m] \to [n]$ and $\alpha^* : X_n \to X_m$, the map $(\mathbf{Z}[X])_n = \mathbf{Z}[X_n] \to \mathbf{Z}[X_m] = (\mathbf{Z}[X])_m$ is given by $\sum_{s \in X_n} n_s s \mapsto \sum_{s \in X_n} n_s \alpha^*(s)$. In particular, for $\alpha = \delta_k$, the face maps of $\mathbf{Z}[X]$ are the maps

$$\mathbf{Z}[X]_n \to \mathbf{Z}[X]_{n-1}, \sum_{s \in X_n} n_s s \mapsto \sum_{s \in X_n} n_s \delta_k(s).$$

Example 3.7. In low degrees, the simplicial abelian group $\mathbf{Z}[\Delta^1]$ is the following:

$$\begin{tikzcd} \dots & \mathbf{Z}_{00} \oplus \mathbf{Z}_{01} \oplus \mathbf{Z}_{11} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbf{Z}_0 \oplus \mathbf{Z}_1 \arrow[l, bend right, "s_0"'] \end{tikzcd}$$

Here the subscripts indicate which element of $(\Delta^1)_k$ the copy of $\mathbf{Z}$ belongs to. The maps d_0 etc. are induced from the corresponding maps on Δ^1. If we write e_{ij} for the element 1 in the copy $\mathbf{Z}_{ij}$ (which is a basis vector), we have that d_0 is the unique $\mathbf{Z}$-linear map sending e_{ij} to e_j, i.e.,

$$d_0(n_{00}e_{00} + n_{01}e_{01} + n_{11}e_{11}) = n_{00}e_0 + (n_{01} + n_{11})e_1$$

and similarly for d_1, s_0 and also the maps in higher dimension.

Definition and Lemma 3.8. Let $X \in \mathrm{sAb}$ be a simplicial abelian group. Then the groups X_n for $n \geqslant 0$ and $X_n := 0$ for $n < 0$ and the following maps constitute a chain complex, denoted $C(X)$:

$$\partial_n : X_n \to X_{n-1}, x \mapsto \sum_{k=0}^{n} (-1)^k d_k(x).$$

The datum $X \mapsto C(X)$ is a functor, called the *chain complex functor*

$$C : \mathrm{sAb} \to \mathrm{Ch}.$$

Proof. We have to check $X_n \xrightarrow{\partial} X_{n-1} \xrightarrow{\partial} X_{n-2}$ vanishes. Let $x \in X_n$. For notational simplicity, we write $x_l := d_l(x) \in X_{n-1}$ and $x_{k,l} := d_k d_l(x) \in X_{n-2}$, for appropriate k, l. Below, we will use the simplicial identity

$$d_k d_l = d_{l-1} d_k$$

for $k < l$, as in (2.2). This gives

$$x_{k,l} = x_{l-1,k}.$$

With this in hand, we have

$$\begin{aligned}
\partial\partial x &:= \partial(\sum_{l=0}^{n}(-1)^l x_l) \\
&= \sum_l (-1)^l \partial x_l \quad (\partial \text{ is a group homomorphism }) \\
&:= \sum_{k=0}^{n-1}\sum_{l=0}^{n}(-1)^k(-1)^l x_{k,l} \\
&= \sum_{l\leqslant k\leqslant n-1}(-1)^{k+l}x_{k,l} + \sum_{k<l\leqslant n}(-1)^{k+l}x_{k,l} \quad \text{(group the terms)} \\
&= \sum_{l\leqslant k\leqslant n-1}(-1)^{k+l}x_{k,l} + \sum_{k<l\leqslant n}(-1)^{k+l}x_{l-1,k} \quad \text{(simplicial identity)} \\
&= \sum_{l<k\leqslant n}(-1)^{k-1+l}x_{k-1,l} + \sum_{l<k\leqslant n}(-1)^{l+k}x_{k-1,l} \quad \text{(rewrite)} \\
&= 0.
\end{aligned}$$

Definition and Lemma 3.9. Let again $X \in \mathrm{sAb}$ be a simplicial abelian group. Let $X_n^{\deg} \subset X_n = C_n(X)$ be the subgroup generated by degenerate simplices. Then the differentials $\partial_n : C_n(X) \to C_{n-1}(X)$ respect this subgroup, so that the groups

$$N(X)_n := C_n(X)/X_n^{\deg},$$

and differentials induced from $C(X)$, constitute a chain complex $N(X)$ called the *normalized chain complex*.

This gives a functor, called the *normalized chain complex functor*

$$N : \mathrm{sAb} \to \mathrm{Ch}$$

such that $N(X)_n = C_n(X)/X_n^{\deg}$, and with differentials induced from $C(X)$.

Proof. Indeed, by the simplicial identities (2.3), modulo degenerate simplices, we have

$$\sum(-1)^k d_k s_j = (-1)^j d_j s_j + (-1)^{j+1} d_{j+1} s_j = 0.$$

The purpose of introducing $N(X)$ is that it is smaller, and therefore more easily useable for concrete computations, than $C(X)$.

However, according to Exercise 4.6, as far as the homology is concerned (which is all that matters here), there is no essential difference between these two complexes.

Example 3.10.

- $C_n(\mathbf{Z}[\Delta^0]) = \mathbf{Z}$, and

$$C(\mathbf{Z}[\Delta^0]) = [\dots \mathbf{Z} \stackrel{0}{\to} \mathbf{Z} \stackrel{\mathrm{id}}{\to} \mathbf{Z} \stackrel{0}{\to} \mathbf{Z} \to 0 \to \dots].$$

  Since the only-non degenerate simplex in Δ^0 is in dimension 0, the normalized complex is just given by $N(\mathbf{Z}[\Delta^0]) = \mathbf{Z}$, concentrated in degree 0.

- For any simplicial set X, $N_n(\mathbf{Z}[X])$ is the free abelian group generated by the non-degenerate n-simplices in X. (Indeed, $X_n = X_n^{\mathrm{deg}} \sqcup X_n^{\mathrm{non-degenerate}}$, which gives $C(\mathbf{Z}[X])_n = \mathbf{Z}[X_n] = \mathbf{Z}[X_n^{\mathrm{deg}}] \oplus \mathbf{Z}[X_n^{\mathrm{non-degenerate}}]$. The degenerate simplices in $C(\mathbf{Z}[X])_n$ are exactly the ones in the first summand.)

- $N(\mathbf{Z}[\Delta^1]) = [\mathbf{Z}_{01} \stackrel{(-1,1)}{\to} \mathbf{Z}_0 \oplus \mathbf{Z}_1]$ (in degrees 1 and 0, the subscripts indicate the basis vectors corresponding to the copies of $\mathbf{Z}$).

- $$N(\mathbf{Z}[\Delta^2]) = \left[\mathbf{Z}_{012} \stackrel{(1,-1,1)}{\to} \mathbf{Z}_{01} \oplus \mathbf{Z}_{02} \oplus \mathbf{Z}_{12} \xrightarrow{\begin{pmatrix} -1 & -1 & \\ 1 & & -1 \\ & 1 & 1 \end{pmatrix}} \mathbf{Z}_0 \oplus \mathbf{Z}_1 \oplus \mathbf{Z}_2 \right].$$

- For the simplicial sphere (Exercise 2.7) we get

$$N(\mathbf{Z}[S^n]) = [\mathbf{Z} \to 0 \dots \to 0 \to \mathbf{Z}] \tag{3.11}$$

  with $\mathbf{Z}$ in degrees n and 0, and all differentials are zero (also for $n = 1$).

3.3 From chain complexes to abelian groups: homology

Definition 3.12. Let C be a chain complex. We define the *cycle groups*, *boundary groups* and *homology groups* of C to be

$$\begin{aligned} Z_n(C) &:= \ker(C_n \stackrel{\partial_n}{\to} C_{n-1}), \\ B_n(C) &:= \operatorname{im}\,(C_{n+1} \stackrel{\partial_{n+1}}{\to} C_n), \\ \mathrm{H}_n(C) &:= Z_n(C)/B_n(C). \end{aligned}$$

Note here that $B_n(C) \subset Z_n(C)$ by definition of a chain complex: for $c \in B_n(C)$, i.e., $c = \partial c'$ for some $c' \in C_{n+1}$ we have $\partial c = \partial\partial c' = 0$, so that $c \in Z_n(C)$. Thus, the homology group is well-defined.

C is called *exact* or *acyclic* if all $\mathrm{H}_n(C) = 0$.

Example 3.13. For a complex C of the form

$$\ldots \to 0 \to C_1 \stackrel{\partial_1}{\to} C_0 \to 0 \to \ldots,$$

we have $\mathrm{H}_1(C) = \ker \partial_1$ and $\mathrm{H}_0(C) = \operatorname{coker} \partial_1$, and all other homologies vanish. This applies, for example, to the complex considered in §1

$$\mathbf{Z}a \oplus \mathbf{Z}b \oplus \mathbf{Z}c \xrightarrow{\begin{pmatrix} -1 & & 1 \\ 1 & -1 & \\ & 1 & -1 \end{pmatrix}} \mathbf{Z}x \oplus \mathbf{Z}y \oplus \mathbf{Z}z.$$

In fact, this complex is $N(\mathbf{Z}[\partial\Delta^2])$, as one can quickly check(!).

Remark 3.14. The same concepts also apply to cochain complexes, except that the differentials go up in degree. Thus, for a cochain complex C, the *cocycles*, *coboundaries* and the all-important *cohomology* groups are defined as

$$\begin{aligned} Z^n(C) &:= \ker(C^n \stackrel{\partial^n}{\to} C^{n+1}), \\ B^n(C) &:= \operatorname{im}\,(C^{n-1} \stackrel{\partial^{n-1}}{\to} C^n), \\ \mathrm{H}^n(C) &:= Z^n(C)/B^n(C). \end{aligned}$$

We will later study cohomology of topological spaces in some depth. Another important example of cohomology arises in analysis when studying the de Rham complex, (3.5). The equation $d^2 = 0$ means that any differential form $\omega \in \Omega^k(M)$ that is exact ($\omega = d\alpha$, for some $k-1$-form α) is also closed ($d\omega = 0$). This raises the question whether the converse holds: is any closed form also exact? The answer to this question depends on M. For example, one shows that for $M = \mathbf{R}^n$, this does hold, which leads to a computation

$$\mathrm{H}^n(\Omega^*(\mathbf{R}^n)) = \begin{cases} \mathbf{R} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

By contrast, for $M = \mathbf{R}^2 \backslash \{0\}$, there is the closed 1-form $\frac{-ydx+xdy}{x^2+y^2} = \frac{dz}{z}$ (for $z = x + iy$, $dz = dx + idy$) which fails to be exact. One shows that the de Rham cohomology is

$$\mathrm{H}^1(\Omega^*(\mathbf{R}^2 \backslash \{0\})) = \mathbf{R}\frac{-ydx + xdy}{x^2 + y^2},$$

i.e., up to multiplication with a scalar $\lambda \in \mathbf{R}$, this form is the only closed, but non-exact 1-form. Later on in this course, we will compute the singular cohomology, which is solely based on topological, not analytical methods:

$$\mathrm{H}^n(\mathbf{R}^2 \backslash \{0\})) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

The so-called *de Rham theorem* asserts that for any (real differentiable) manifold M, there is a canonical isomorphism

$$\mathrm{H}^n(M, \mathbf{R}) = \mathrm{H}^n(\Omega^*(M)).$$

This displays a substantial link between the topology of some space and the solvability of differential equations. See, for example, [War83] for all of this.

Lemma 3.15. The cycle, boundary and homology groups are functors

$$Z_n, B_n, \mathrm{H}_n : \mathrm{Ch} \to \mathrm{Ab}.$$

Proof. Left as an exercise (!)(you will need to use the commutativity of (3.2) at some point). ! □

3.4 Homology of simplicial sets: definition and examples

Definition 3.16. We define the homology groups of a simplicial set X as

$$\mathrm{H}_n(X) := \mathrm{H}_n(N(\mathbf{Z}[X])).$$

The homology functor is the composition

$$\mathrm{sSet} \stackrel{\mathbf{Z}[-]}{\to} \mathrm{sAb} \stackrel{N}{\to} \mathrm{Ch} \stackrel{\mathrm{H}_n}{\to} \mathrm{Ab}.$$

More generally, for any commutative ring Λ, we define

$$\mathrm{H}_n(-, \Lambda)$$

similarly, by replacing $\mathbf{Z}[-]$ above by $\Lambda[-]$, i.e., we take the free Λ-module generated by the n-simplices of X. We refer to this as *homology with* Λ*-coefficients*. Thus $\mathrm{H}_n(X) = \mathrm{H}_n(X, \mathbf{Z})$. The rôle of Λ is the following: we will eventually prove the *universal coefficient theorem* (todo: ref todo]ref) which states that

- $\mathrm{H}_n(X, \mathbf{Q}) = \mathrm{H}_n(X) \otimes_{\mathbf{Z}} \mathbf{Q}$, so homology with rational coefficients just forgets about the torsion part in the groups $\mathrm{H}_n(X)$, which makes it sometimes easier to compute.
- As far as torsion is concerned, $\mathrm{H}_n(X, \mathbf{Z}/\ell)$ will be a mixture of $\mathrm{H}_n(X)/\ell$ and $\{a \in \mathrm{H}_{n-1}(X), \ell a = 0\}$, i.e., the ℓ-torsion in the $n-1$-st homology group. Thus, homology with torsion coefficients can, in some cases, detect finer phenomena than $\mathrm{H}_n(X, \mathbf{Z})$. See Exercise 4.5 for a precise statement. An interesting example for homology with torsion coefficients appears in Example 3.20.

Example 3.17. From Example 3.10, we get the following computations:

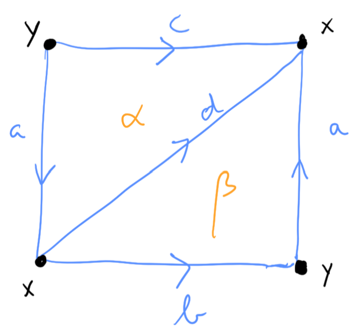
- For the k-simplex, we get

  $$\mathrm{H}_n(\Delta^k) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

  For $k = 0$ this is immediate, and for $k = 1, 2$, it follows by inspection of the normalized chain complexes (Exercise 3.6). For $k > 2$, we will use a more convenient method, homotopies, below in Exercise 3.7.

- For the simplicial k-sphere S^k, the complex in (3.11) immediately gives
$$\mathrm{H}_n(S^k) = \begin{cases} \mathbf{Z} & n = 0, n = k \\ 0 & \text{otherwise} \end{cases}$$
Note that the right hand side agrees exactly with the claim made for the homology of the *topological* k-sphere S^k_{top} (cf. also the comments made after Example 2.18). This agreement is not a coincidence: as a consequence of excision, we will eventually compute the homology of S^k_{top} by showing that it agrees with the one of the simplicial k-sphere S^k.

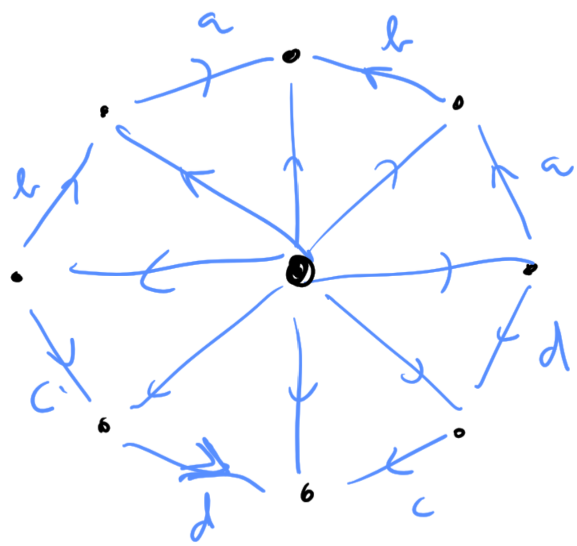
Example 3.18. We consider the *Möbius strip* M:

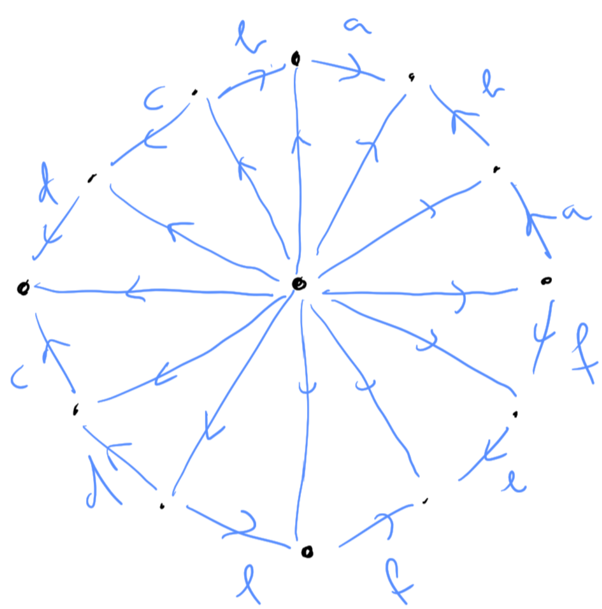


According to Exercise 2.9, the inclusion
$$S^1 \xrightarrow{i} M$$
is a homotopy equivalence. We will show in Corollary 3.42 that this implies that
$$i_* : \mathrm{H}_k(S^1) \to \mathrm{H}_k(M)$$
is an isomorphism. Thus
$$\mathrm{H}_k(M) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$
A generator of $\mathrm{H}_1(M)$ is given by (the class of) the edge d. To see $\mathrm{H}_2(M) = 0$, note that $n_\alpha \alpha + n_\beta \beta$ is mapped under the differential to
$$n_\alpha(d - c + a) + n_\beta(a - d + b) = (n_\alpha + n_\beta)a + n_\beta b - n_\alpha c + (n_\alpha - n_\beta)d,$$

which is zero only for $n_\alpha = n_\beta = 0$, so that the cycle group Z^2 vanishes, and a fortiori $\mathrm{H}_2(M)$.

Example 3.19. Let $g \geqslant 1$. We define a simplicial set X_g to be glued from $4g$ copies of Δ^2, with boundaries identified as shown for $g = 2$ and $g = 3$, respectively:





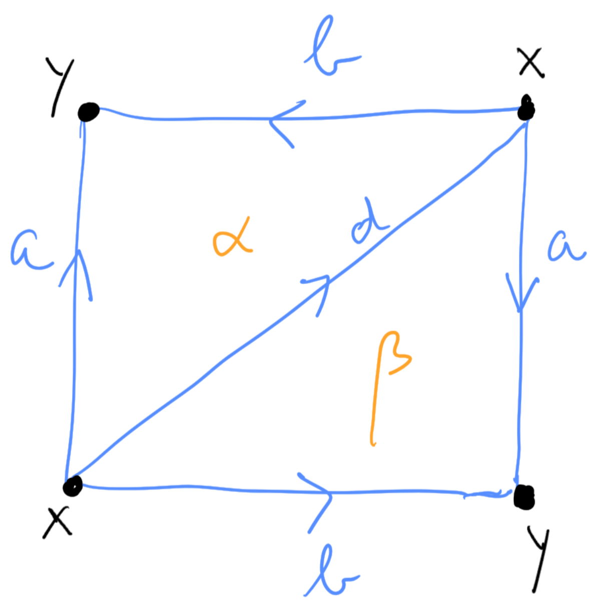
This simplicial set is called the (simplicial) *orientable surface with genus g*, since its geometric realization $|X_g|$ has (up to homeomorphism) precisely that property. The *genus* is, roughly speaking, the number of handles attached to S^2_{Top}. We will shortly see how the genus arises from the computation of the homology of X_g. An animation that shows the construction in case $g = 2$ is found here: https://youtu.be/G1yyfPShgqw.

Then, we have

$$\mathrm{H}_k(X_g) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z}^{2g} & n = 1 \\ \mathbf{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

A basis of $\mathrm{H}_1(X_g)$ is given by the outer edges (e.g., a, b, c, d, e, f in case $g = 3$). The fact that the ranks of these groups are symmetric ($\mathrm{rk}\,\mathrm{H}_k = \mathrm{rk}\,\mathrm{H}_{2-k}$) is no coincidence, but rather an example of so-called *Poincaré duality*, which asserts that this symmetry holds for any compact orientable manifold.

Example 3.20. We compute the homology of the (simplicial) projective plane P^2, which is the simplicial set pictured as follows:



Unlike for the examples before, it becomes interesting to not only consider homology with $\mathbf{Z}$-coefficients, but general rings Λ. P^2 has the following non-degenerate simplices: α, β (in dimension 2), the edges a, b, d and the vertices x, y. We then get the following chain complex

$$\Lambda_\alpha \oplus \Lambda_\beta \xrightarrow{\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}} \Lambda_a \oplus \Lambda_b \oplus \Lambda_d \xrightarrow{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}} \Lambda_x \oplus \Lambda_y.$$

(The first column of the left hand matrix reflects the fact that $\partial(\alpha) = d_0(\alpha) - d_1(\alpha) + d_2(\alpha) = b - a + d$, for the right hand matrix note that $\partial(a) = y - x$ etc.) (The first column of the left hand matrix reflects the fact that $\partial(\alpha) = d_0(\alpha) - d_1(\alpha) + d_2(\alpha) = b - a + d$, for the right hand matrix note that $\partial(a) = y - x$ etc.)

- For degree 0, we have $B_0 = \{(x, -x), x \in \Lambda\}$, so that

$$\mathrm{H}_0(P^2, \Lambda) = \Lambda.$$

- For degree 1, we have $Z_1 = \{(x_a, x_b, x_d), x_a + x_b = 0\}$, while B_1 is the image of the 2-by-3 matrix displayed above, which is the same as the image of the 2-by-3 matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 2 & 1 \end{pmatrix}.$$

  We have an isomorphism

$$\begin{aligned} \mathrm{H}_1(P^2, \Lambda)(:= Z_1/B_1) &\to \Lambda/2, \\ (x_a, x_b, x_d) &\mapsto x_d - x_a. \end{aligned}$$

  Indeed, this map is clearly surjective. It is also injective: for $x_d - x_a \in 2\Lambda$, say $x_d - x_a = 2x$ for some $x \in \Lambda$ (and $x_a + x_b = 0$), we have $(x_a, x_b, x_d) = (x_a, -x_a, x_a + 2x) \in B_1 = \operatorname{im} \partial_2$.

- For degree 2, we have

$$\mathrm{H}_2(P^2) = Z_2 = \{(x_a, x_b), x_\alpha + x_\beta = 0, x_\alpha - x_\beta = 0\} \cong \{x \in \Lambda, 2x = 0\}. \blacksquare$$

- Of course, in higher degrees, $\mathrm{H}_k(P^2) = 0$.

We therefore see that the homology depends in an interesting way on Λ:

$$\mathrm{H}_k(P^2, \mathbf{Z}) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z}/2 & n = 1 \\ 0 & n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{H}_k(P^2, \mathbf{Z}/2) = \begin{cases} \mathbf{Z}/2 & n = 0 \\ \mathbf{Z}/2 & n = 1 \\ \mathbf{Z}/2 & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

3.5 Chain homotopies

In §2.4 we introduced the notion of continuous and simplicial homotopies, which are special relations between two maps

$$f, g : X \to Y$$

in Top, and in sSet, respectively. In this section, we are going to define the corresponding notion of homotopies between maps of chain complexes. We will show that

- if $r, s : C \to D$ are two maps of chain complexes that are chain homotopic, then they induce the *same* map

  $$\mathrm{H}_n(r) = \mathrm{H}_n(s)$$

  on homology groups,

- if $f, g : X \to Y$ are (simplicially) homotopic maps in sSet, then the maps $N(f), N(g) : N(X) \to N(Y)$ are chain homotopic. Here and in the sequel, we will sometimes abbreviate

  $$N(X) := N(\mathbf{Z}[X]), N(f) := N(\mathbf{Z}[f]).$$

These facts are the key ingredients in the homotopy invariance of homology (Proposition 4.8).

Definition 3.21. • Let $f, g : C \to D$ be two chain maps. A *chain homotopy* between f and g is a collection of group homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$\partial^D_{n+1} \circ h_n + h_{n-1} \circ \partial^C_n = g_n - f_n. \tag{3.22}$$

This condition is often written as

$$\partial h + h\partial = g - f.$$

More pictorially, the sum of the two composites in the lozenge must equal the vertical maps:

$$\begin{tikzcd}
 & \cdots \arrow[r] & C_n \arrow[r, "\partial_n^C"] \arrow[dl, "h_n"'] \arrow[d, "g_n - f_n"] & C_{n-1} \arrow[r] \arrow[dl, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r] & \cdots &
\end{tikzcd}$$

- We say that $f : C \to D$ is (chain) homotopic to $g : C \to D$ if there is such a chain homotopy.
- A chain map $f : C \to D$ is called a *chain homotopy equivalence* if there is a chain map $g : D \to C$ such that $f \circ g$ is homotopic to id_D, and $g \circ f$ is chain homotopic to id_C.

Remark 3.23. Note that the h_n do *not* assemble to a chain map $C \to D[1]$! Instead, we will soon see that a homotopy is the same thing as a chain map

$$N(\mathbf{Z}[\Delta^1]) \otimes C = (\mathbf{Z} \overset{(-1,1)}{\to} \mathbf{Z} \oplus \mathbf{Z}) \otimes C \to D.$$

Lemma 3.24. Let $f, g : C \to D$ be two homotopic chain maps. Then $\mathrm{H}_n(f) = \mathrm{H}_n(g)$ for all $n \in \mathbf{Z}$.

Proof. Let $c \in Z_n(C)$ be a cycle. We have to show that $g(c) - f(c)$ is a boundary (in D), so that $[g(c)] - [f(c)] = 0 \in \mathrm{H}_n(D)$. Indeed:

$$g(c) - f(c) = \partial h(c) + h \underbrace{\partial(c)}_{=0} \in B_n(D).$$

Definition 3.25. A chain map $f : C \to D$ is called a *quasi-isomorphism* if the induced maps

$$\mathrm{H}_n(f) : \mathrm{H}_n(C) \to \mathrm{H}_n(D)$$

is an isomorphism for each $n \in \mathbf{Z}$.

We say a complex C is *quasi-isomorphic* to a complex D if there is a quasi-isomorphism $f : C \to D$.

Lemma 3.26. Any chain homotopy equivalence $f : C \to D$ is a quasi-isomorphism.

Proof. Indeed, for $g : D \to C$ as above, the following maps $\mathrm{H}_n(D) \to \mathrm{H}_n(D)$ agree:

$$\mathrm{H}_n(f) \circ \mathrm{H}_n(g) = \mathrm{H}_n(f \circ g) = \mathrm{H}_n(\mathrm{id}_D) = \mathrm{id}_{\mathrm{H}_n(D)}$$

by Lemma 3.15 and Lemma 3.24. Similarly with $g \circ f$, so that $\mathrm{H}_n(f)$ is an isomorphism. □

Synopsis 3.27. Given a chain map $f : C \to D$, we can list a number of conditions, where each one implies the one below:
(1) f is an isomorphism
(2) f is a chain homotopy equivalence
(3) f is a quasi-isomorphism

Remark 3.28. In general, none of these implications is reversible.

- (3) $\not\Rightarrow$ (2): for example, the map of chain complexes
$$C := [0 \to \mathbf{Z} \xrightarrow{e} \mathbf{Z} \to 0 \ldots] \xrightarrow{f} D := \mathbf{Z}/e[0]$$
(given in degree 0 by the canonical projection) is a quasi-isomorphism: the homology groups H_n of both complexes vanish for $n \neq 0$. We have $Z_0(C) = \mathbf{Z}$, $B_0(C) = e\mathbf{Z}$, so that $\mathrm{H}_0(C) = \mathbf{Z}/e\mathbf{Z}$. The induced map $\mathrm{H}_0(C) \to \mathrm{H}_0(D)$ is the identity, so f is a quasi-isomorphism. However, there is no nonzero group homomorphism $\mathbf{Z}/e \to \mathbf{Z}$, so any $g : D \to C$ must be zero. However, $f \circ 0 = 0$ is not homotopic to id_D: this would imply that the 0-map and the identity of $\mathrm{H}_0(D)$ agree, which is false.
- However, it can be shown that if C and D are complexes of modules over a ring Λ such that $C_n = D_n = 0$ for $n \ll 0$, and all C_n and D_n are *projective* Λ-modules, then a quasi-isomorphism f is necessarily a chain homotopy equivalence. See, e.g., [Wei94, Theorem 2.2.6] and related statements there.
- If one has two chain complexes C and D such that for all n, there are (group) isomorphisms $e_n : \mathrm{H}_n(C) \xrightarrow{\cong} \mathrm{H}_n(D)$, C need not be quasi-isomorphic to D: there need not be a chain map

$f : C \to D$ such that the induced maps $\mathrm{H}_n(f)$ agree with e_n. (However, if such a map f does exist, then it is of course a quasi-isomorphism).

The relation "C is quasi-isomorphic to D" is not symmetric: given a quasi-isomorphism $f : C \to D$, there need not be a quasi-isomorphism $D \to C$, cf. Exercise 3.3.

3.6 From simplicial homotopies to chain homotopies

To establish a connection between simplicial homotopies and chain homotopies, we introduce the tensor product of chain complexes, and show that the normalized chain complex functor N behaves well in this regard (Proposition 3.41). The tensor product of chain complexes will also play a key role in the Künneth formula that computes the cohomology of a product of two topological spaces ($\boxed{\text{todo: link}}$todo]link).

3.6.1 Tensor products of chain complexes

Definition and Lemma 3.29. The *tensor product* of two chain complexes $C, D \in \mathrm{Ch}$ is defined to be the complex with $(C \otimes D)_k = \bigoplus_{m+n=k} C_m \otimes D_n$. The differential

$$\partial_k^{C \otimes D} : (C \otimes D)_k \to (C \otimes D)_{k-1}$$

is defined by the formula

$$\partial(c \otimes d) := (\partial c) \otimes d + (-1)^m c \otimes (\partial d), \tag{3.30}$$

where $c \in C_m$, $d \in D_n$.

This is indeed a chain complex, which gives a functor

$$\otimes : \mathrm{Ch} \times \mathrm{Ch} \to \mathrm{Ch}.$$

Proof. Using $\partial_C^2 = \partial_D^2 = 0$, we compute

$$\begin{aligned} \partial(\partial(c \otimes d)) &= \partial\left((\partial c) \otimes d + (-1)^m c \otimes (\partial d)\right) \\ &= \underbrace{\partial(\partial c)}_{=0} \otimes d + (-1)^{m-1}(\partial c) \otimes (\partial d) + (-1)^m (\partial c) \otimes (\partial d) + (-1)^m (-1)^m c \otimes \underbrace{\partial(\partial}_{=} \\ &= 0. \end{aligned}$$

Example 3.31.
- For any complex C and any abelian group Λ, the complex $C \otimes (\Lambda[0])$ is given in degree n by $C_n \otimes \Lambda$, with differential $\partial_C \otimes \mathrm{id}$. In particular, for a simplicial set X,
$$N(\mathbf{Z}[X]) \otimes \Lambda = N(\Lambda[X]).$$
- For $C \in \mathrm{Ch}$, the shifted complex (Example 3.4) is given by
$$C[n] = (\mathbf{Z}[n]) \otimes C.$$
Indeed, $\mathbf{Z}[n]_k = \mathbf{Z}$ for $k = -n$ and 0 otherwise, so that $(\mathbf{Z}[n] \otimes C)_k = C_{n+k}$. The differential is given by
$$\partial_{\mathbf{Z}[n]\otimes C}(s \otimes c) = \underbrace{\partial_{\mathbf{Z}[n]}(s)}_{=0} \otimes c + (-1)^{-n} s \otimes \partial_C c = (-1)^n s \partial_C c.$$
Here $s \in \mathbf{Z}[n]_{-n} = \mathbf{Z}$, and $c \in C_{n+k}$. By contrast, $C \otimes \mathbf{Z}[n]$ has again C_{n+k} in degree k, and no sign in the differential:
$$\partial_{C\otimes\mathbf{Z}[n]}(c \otimes s) = \partial c \otimes s + (-1)^{n+k} c \otimes \partial s = \partial c \otimes s.$$
Thus, the maps
$$f_k := (-1)^n \mathrm{id} : (C \otimes \mathbf{Z}[n])_k = C_{n+k} \to C_{n+k} = (\mathbf{Z}[n] \otimes C)_k$$
constitute an isomorphism of chain complexes (which is not the identity!).

Lemma 3.32. Let $f, g : C \to D$ be two chain maps. A chain homotopy h between f and g (Definition 3.21) is the same thing as a chain map h fitting into a commutative diagram like so:

$$\begin{tikzcd}
N(\Delta^0) \otimes C \arrow[r, equal] \arrow[d, "\delta_0 \otimes \mathrm{id}"'] & C \arrow[dr, "f"] & \\
N(\Delta^1) \otimes C \arrow[rr, "h"] & & D \\
N(\Delta^0) \otimes C \arrow[r, equal] \arrow[u, "\delta_1 \otimes \mathrm{id}"] & C. \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. By Example 3.10, $N(\Delta^1) = [\mathbf{Z} \stackrel{(-1,1)}{\to} \mathbf{Z} \oplus \mathbf{Z}]$, where the left hand $\mathbf{Z}$ is in degree 1. Thus,
$$(N\Delta^1 \otimes C)_n = \mathbf{Z} \otimes C_{n-1} \otimes (\mathbf{Z} \oplus \mathbf{Z}) \otimes C_n = C_{n-1} \oplus C_n \oplus C_n.$$

The restriction along $\delta_k : \Delta^0 \to \Delta^1$ being f and g, respectively, only leaves free the maps $C_{n-1} \to D_n$, which we call h_n. Unwinding the definition of the differential on the tensor product, one checks (!)that h is a chain map iff the above diagram commutes holds. □

3.6.2 The Eilenberg–Zilber map

In this section we show that the functor

$$N(\mathbf{Z}[-]) : \mathrm{sSet} \to \mathrm{Ch}$$

interacts well with the monoidal structures, i.e., relates products in sSet with tensor products in Ch. This is then used in order to show that simplicial homotopies give rise to chain homotopies under that functor.

Following up on Example 2.25, we begin with a closer look at the product of simplicial sets $\Delta^n \times \Delta^m$.

Lemma 3.33. Let $m, n \geqslant 0$. Then the following holds:

(1) A p-simplex of $\Delta^m \times \Delta^n$ corresponds to a pair of morphisms (in the category Δ)

$$(\sigma_1 : [p] \to [m], \sigma_2 : [p] \to [n])$$

or, equivalently, to an order-preserving map

$$\sigma : [p] \to [m] \times [n],$$

(where at the right hand we declare $(i, j) \leqslant (i', j')$ iff $i \leqslant i'$ and $j \leqslant j'$).

(2) A p-simplex is non-degenerate iff the map σ is injective. The highest possible p with that property is $p = m + n$.

(3) Any non-degenerate simplex of $\Delta^m \times \Delta^n$ is a face of (i.e., arises by applying appropriate face maps to) a non-degenerate $(m+n)$-simplex.

(4) There is a bijection between

$$(\Delta^m \times \Delta^n)^{\mathrm{non-deg}}_{m+n} \cong \{J \subset \{1, \dots, m+n\}, |J| = m\}. \tag{3.34}$$

At the right, we have the subsets of $[m+n]$ with cardinality m.

Proof. The first statement holds by Lemma 2.33. For the second, observe that σ is injective iff it does not factor over some map $\sigma_k : [p] \to [p-1]$ iff it is non-degenerate. (3) is left as an exercise.

Finally, take an injective order-preserving map

$$\sigma = (\sigma_1, \sigma_2) : [m+n] \to [m] \times [n]$$

and assign to it the subset

$$J := \{1 \leqslant j \leqslant m+n, \sigma_1(j-1) = \sigma_1(j)\} (= \{1 \leqslant j \leqslant m+n, \sigma_2(j-1) < \sigma_2(j)\}).$$

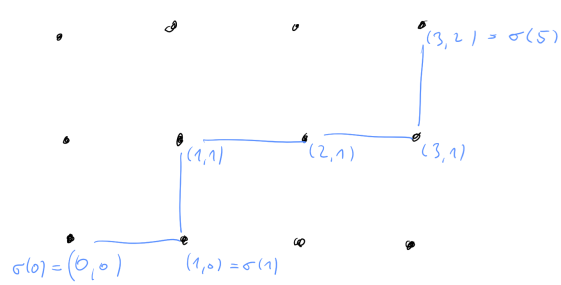
Since σ is injective and order-preserving, $|J| = m$, and one checks it defines a bijection. □

Definition 3.35. A subset $J \subset \{1, \ldots, m+n\}$, $|J| = m$ is called a *shuffle*, with the idea that an (ordered) deck of m cards is shuffled into an (ordered) deck of n cards, without changing the order within the two decks.

More formally, J gives rise to a unique permutation of the set $\{1, \ldots, m+n\}$ such that $1, \ldots, m$ map to the elements in J in the order-preserving way, and the elements $m+1, \ldots, n$ to $[n+m] \backslash J$, again in the order-preserving way.

The *signature* of a shuffle $\mathrm{sgn}(J)$ is defined as the signature of that permutation.

Example 3.36. To the displayed map $\sigma : [5] \to [3] \times [2]$ corresponds the shuffle $J = \{2, 5\}$, which gives rise to the permutation 31452, whose signature is $\mathrm{sgn}(J) = (-1)^4 = +1$.



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• 1

• 4

• 5

• 2

Let $X, Y \in \mathrm{sSet}$. In order to define a chain map

$$N(\mathbf{Z}[X]) \otimes N(\mathbf{Z}[Y]) \to N(\mathbf{Z}[X \times Y]),$$

recall that the k-simplices are the free abelian groups

$$\begin{aligned}(N(\mathbf{Z}[X]) \otimes N(\mathbf{Z}[Y]))_k &= \bigoplus_{m+n=k} N(\mathbf{Z}[X])_m \otimes N(\mathbf{Z}[Y])_n \\ &= \mathbf{Z}[X_m^{\mathrm{non-deg}}] \otimes \mathbf{Z}[Y_n^{\mathrm{non-deg}}] \\ &= \mathbf{Z}[X_m^{\mathrm{non-deg}} \times Y_n^{\mathrm{non-deg}}],\end{aligned}$$

so to specify such a map, we need to send any pair (α, β) of non-degenerate simplices (in X and Y, respectively), to an element in

$$N(\mathbf{Z}[X \times Y])_k,$$

i.e., a formal linear combination of non-degenerate simplices in $X \times Y$.

Recall also from Lemma 2.33, that $X_m \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^m, X)$. Using this, we can further constrain the way how to construct such maps:

$$\begin{tikzcd} \Delta^{m+n} \arrow[r, "\sigma"] & \Delta^m \times \Delta^n \arrow[d, "\alpha\times\beta"] \\ & X \times Y. \end{tikzcd} \tag{3.37}$$

I.e., given α and β, we can use a shuffle σ (i.e., a non-degenerate top-dimensional simplex of $\Delta^m \times \Delta^n$), and get an $(m+n)$-simplex of $X \times Y$.

Definition 3.38. The *Eilenberg–Zilber map* is the map

$$\nabla : N(\mathbf{Z}[X]) \otimes N(\mathbf{Z}[Y]) \to N(\mathbf{Z}[X \times Y]),$$

that assigns to a pair of non-degenerate simplices (α, β) as above the sum

$$\sum_{J \text{ shuffle}} \operatorname{sgn}(J)((\alpha \times \beta) \circ \sigma_J),$$

where σ_J corresponds to J under the bijection in (3.34).

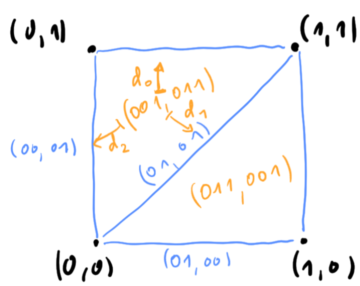
Example 3.39. Let us illustrate the definition of ∇ with the example $X = Y = \Delta^1$. Recall that

$$N(\Delta^1) = [\mathbf{Z}_{01} \overset{(-1,1)}{\to} \mathbf{Z}_0 \oplus \mathbf{Z}_1],$$

so that (as usual, subscripts serve to remember the generators and $\mathbf{Z}_{?\otimes\bullet}$ stands for $\mathbf{Z}_? \otimes \mathbf{Z}_\bullet$):

$$N(\Delta^1)\otimes N(\Delta^1) = [\mathbf{Z}_{01\otimes 01} \to \mathbf{Z}_{01\otimes 0}\oplus\mathbf{Z}_{01\otimes 1}\oplus\mathbf{Z}_{0\otimes 01}\oplus\mathbf{Z}_{1\otimes 01} \to \bigoplus_{i,j=0}^{1} \mathbf{Z}_{i\otimes j}].$$

By comparison, $\Delta^1 \times \Delta^1$ has the following simplices (cf. Example 2.25)



so that

$$N(\Delta^1\times\Delta^1) = [\mathbf{Z}_{001,011}\oplus\mathbf{Z}_{011,001} \to \mathbf{Z}_{00,01}\oplus\mathbf{Z}_{01,11}\oplus\mathbf{Z}_{01,01}\oplus\mathbf{Z}_{01,00}\oplus\mathbf{Z}_{11,01} \to \bigoplus_{i,j=0}^{1} \mathbf{Z}_{i,j}].$$

- In degree 0, ∇ is composed of the identity maps $\mathbf{Z}_{i\otimes j} \to \mathbf{Z}_{i,j}$. Indeed, in (3.37), there is exactly one shuffle to be considered, which is $J = \varnothing \subset (\{1, \dots, 0\} = \varnothing$, which corresponds under the bijection in (3.34) to the identity map $\sigma = \mathrm{id} : [0] \to [0] \times [0]$ and therefore the identity permutation (of $\varnothing$), so that the sign is $+1$.

- In degree 1, ∇ is composed by identity maps $\mathbf{Z}_{i\otimes 01} \to \mathbf{Z}_{ii,01}$ and $\mathbf{Z}_{01\otimes i} \to \mathbf{Z}_{01,ii}$. Indeed, focus on the first case (the other is similar, also with sign $+1$): in (3.37) (with $m = 0$ and $n = 1$), there is exactly one shuffle to be considered, namely $J = \varnothing \subset \{1\}$, which corresponds again to the identity map $\sigma = \mathrm{id} : [1] \to [0] \times [1]$, whose signature is again $+1$. Thus,

$$\begin{tikzcd} \Delta^{0+1} \arrow[r, "\sigma"] \arrow[dr, "ii{,}01"'] & \Delta^0 \times \Delta^1 \arrow[d, "i\times 01"] \\ & \Delta^1 \times \Delta^1. \end{tikzcd}$$

- In degree 2, things get more interesting:

$J \subset \{1,2\}$	σ	$\mathrm{sgn}(\sigma)$	$[2] \to [1] \times [1]$	2-simplex in $\Delta^1 \times \Delta^1$
$\{1\}$	12	$+1$	$(0,0),(1,0),(1,1)$	$(011,001)$
$\{2\}$	21	-1	$(0,0),(0,1),(1,1)$	$(001,011)$

Thus, in degree 2, ∇ is the map

$$\mathbf{Z}_{01\otimes 01} \xrightarrow{(-1,+1)} \mathbf{Z}_{001,011} \oplus \mathbf{Z}_{011,001}$$

Lemma 3.40. The Eilenberg–Zilber map ∇ is indeed a chain map.

Proof. We have to check

$$\partial\nabla = \nabla\partial,$$

which we only do in the above example $X = Y = \Delta^1$. The proof in general uses the same idea, but slightly more tedious combinatorical arguments. See, e.g., [Lur, Tag 00RR]. The point is that the signs of the shuffles are such that the diagonal edge $01, 01$, which arises as a face of both non-degenerate 2-simplices in $\Delta^1 \times \Delta^1$, cancel each other: Indeed,

$$\begin{aligned}
\partial\nabla(01 \otimes 01) &= \partial((011,001) - (001,011)) \\
&= ((11,01) - (01,01) + (01,00)) - ((01,11) - (01,01) + (00,01)) \\
&= (11,01) - (01,01) + (01,00) - (01,11) + (01,01) - (00,01) \\
&= (11,01) + (01,00) - (01,11) - (00,01).
\end{aligned}$$

On the other hand, using the signs in the tensor product in (3.30), we have

$$\begin{aligned}
\nabla\partial(01 \otimes 01) &= \nabla\left(\partial(01) \otimes 01 - 01 \otimes \partial(01)\right) \\
&= \nabla\left((1 \otimes 01 - 0 \otimes 01) - (01 \otimes 1 - 01 \otimes 0)\right) \\
&= 11 \otimes 01 - 00 \otimes 01 - 01 \otimes 11 + 01 \otimes 00.
\end{aligned}$$

Here comes the reward:

Proposition 3.41. Let $f, g : X \to Y$ be (simplicially) homotopic maps between simplicial sets. Then the induced maps

$$N(\mathbf{Z}[f]), N(\mathbf{Z}[g]) : N(\mathbf{Z}[X]) \to N(\mathbf{Z}[Y])$$

are chain homotopic. Thus, by Lemma 3.24,

$$\mathrm{H}_n(f), \mathrm{H}_n(g) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

are the *same* maps.

Proof. This is an immediate consequence of the existence of the Eilenberg–Zilber map ∇. We again write $N(X) := N(\mathbf{Z}[X])$ etc. Let $h : \Delta^1 \times X \to Y$ be a homotopy between f and g. Then we have a commutative diagram:

$$\begin{tikzcd}
N(\Delta^0) \otimes N(X) \arrow[r, equal] \arrow[d, "\delta^0"] & N(\Delta^0 \times X) \arrow[r, equal] \arrow[d, "\delta^0"] & N(X) \arrow[dr, "N(f)"] & \\
N(\Delta^1) \otimes N(X) \arrow[r, dotted, "\nabla"] & N(\Delta^1 \times X) \arrow[rr, "N(h)"] & & N(Y) \\
N(\Delta^0) \otimes N(X) \arrow[r, equal] \arrow[u, "\delta^1"] & N(\Delta^0 \times X) \arrow[r, equal] \arrow[u, "\delta^1"] & N(X). \arrow[ur, "N(g)"] &
\end{tikzcd}$$

Indeed, $N(\mathbf{Z}[\Delta^1]) = [\mathbf{Z} \xrightarrow{(-1,1)} \mathbf{Z} \oplus \mathbf{Z}]$, so that for any chain complex C,

$$(N(\mathbf{Z}[\Delta^1]) \otimes C)_k = C_{k-1} \oplus \mathbf{Z}[X_k^{\text{non-deg}}] \oplus \mathbf{Z}[X_k^{\text{non-deg}}] \to \mathbf{Z}[(\Delta^1 \times X)_k^{\text{non-deg}}].$$

Thus,

$$N(h) \circ \nabla : N(\Delta^1) \otimes N(X) \to N(Y)$$

is a chain homotopy between $N(f)$ and $N(g)$. $\square$

The next corollary will be used to compute the homology of some topological spaces:

Corollary 3.42. Any simplicial homotopy equivalence $f : X \to Y$ gives rise to a quasi-isomorphism

$$N(f)(:= N(\mathbf{Z}[f]) : N(X) \to N(Y),$$

and thus to isomorphisms

$$\mathrm{H}_n(f) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y).$$

3.7 Exercises

Exercise 3.1. Prove Lemma 3.15.

For the cycles (but not the others) one may argue by exhibiting a (very small) chain complex Z_n, such that for each chain complex C,

$$Z_n(C) = \mathrm{Hom}_{\mathrm{Ch}}(Z_n, C).$$

Exercise 3.2. Let HoCh be the category of *chain complexes up to homotopy*: its objects are chain complexes, and

$$\mathrm{Hom}_{\mathrm{HoCh}}(C, D) := \mathrm{Hom}_{\mathrm{Ch}}(C, D)/\sim,$$

where $f \sim g$ iff the two maps are chain homotopic.

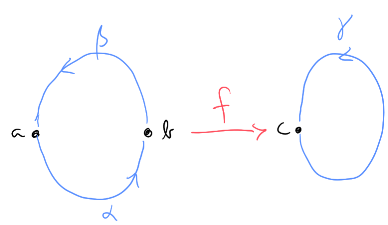
- Verify this is indeed a well-defined category and that there is a functor
$$\mathrm{Ch} \to \mathrm{HoCh}$$
given on objects by $C \mapsto C$.
- Show that $\mathrm{H}_n(C) = \mathrm{Hom}_{\mathrm{HoCh}}(\mathbf{Z}[n], C)$.

Exercise 3.3. Is there a quasi-isomorphism

$$\mathbf{Z}/n \to [\mathbf{Z} \xrightarrow{n} \mathbf{Z}],$$

(where the left complex is concentrated in degree 0, the right one in degrees 1 and 0)?

Exercise 3.4. Let $f : X \to Y$ be the following map of simplicial sets



i.e., $a, b \mapsto c$, $\alpha, \beta \mapsto \gamma$.

- Compute the normalized chain complex of X, of Y and show that the homologies are given by
$$\mathrm{H}_k(X) = \mathrm{H}_k(Y) = \mathbf{Z}$$
for $k = 0$ and $k = 1$.

- Show that, under the isomorphisms above, the map

$$\mathrm{H}_k(f) : \mathrm{H}_k(X) \to \mathrm{H}_k(Y)$$

  is the identity for $k = 0$ and multiplication by 2 for $k = 1$.

- Let $r \in \mathbf{Z}$, $r \geqslant 2$. Describe a map of simplicial sets

$$f^{(r)} : X^{(r)} \to Y,$$

  (for an appropriate $X^{(r)} \in \mathrm{sSet}$) such that $\mathrm{H}_k(X^{(r)}) = \mathrm{H}_k(Y) = \mathbf{Z}$ for $k = 0, 1$ and such that (under these isomorphisms)

$$\mathrm{H}_0(f^{(r)}) = \mathrm{id}_{\mathbf{Z}}, \mathrm{H}_1(f^{(r)}) = r\mathrm{id}_{\mathbf{Z}},$$

  i.e., multiplication by r in the first homology group.

Exercise 3.5. Let C be an exact complex. Show that the following are equivalent:
(1) $\partial_n : C_n \to C_{n-1}$ is 0,
(2) ∂_{n+1} is surjective,
(3) ∂_{n-1} is injective.

Exercise 3.6. Show by direct computation that

$$\mathrm{H}_n(\Delta^2) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

"Draw" some elements in $Z_1(N(\mathbf{Z}[\Delta^2]))$ and $B_1(N(\mathbf{Z}[\Delta^2]))$.

Exercise 3.7. Show that the unique map $\Delta^k \to \Delta^0$ is a simplicial homotopy equivalence. (A simplicial set with this property is called *contractible*.) Use this to confirm the claim made in Example 3.17.

Hint: there is a conceptual proof, which relies on expressing Δ^k as a nerve (cf. Exercise 2.4): $\Delta^k = N([k])$. Now show that for a category C with an initial object, $N(C)$ is contractible. Alternatively, there is also a hands-on proof by writing down simplices of $\Delta^1 \times \Delta^k$.

Exercise 3.8. Using the product of simplicial sets from Definition 2.24, we define the *simplicial torus*

$$T := S^1 \times S^1.$$

- Draw the non-degenerate simplices of T.

- Compute the complex $N(\mathbf{Z}[T])$.
- Show

$$\mathrm{H}_n(T) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z}^2 & n = 1 \\ \mathbf{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

- (If you feel adventurous:) What do the ranks of the groups $\mathrm{H}_n(T)$ look like? Make a guess for $\mathrm{H}_n(S^1 \times S^1 \times S^1)$ and prove it!

Exercise 3.9. Let

$$\ldots \to 0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0 \ldots$$

be an exact complex of abelian groups. (This is called a *short exact sequence*.) Show that for any abelian group T, there are complexes, with appropriate natural maps

$$0 \to \mathrm{Hom}_{\mathrm{Ab}}(T, A) \xrightarrow{a_*} \mathrm{Hom}_{\mathrm{Ab}}(T, B) \xrightarrow{b_*} \mathrm{Hom}_{\mathrm{Ab}}(T, C) \to 0.$$

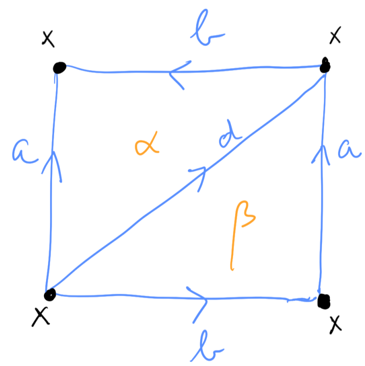
Show that this complex is exact except possibly at the spot $\mathrm{Hom}(T, C)$, i.e., b_* need not be surjective. Show that for a *free* abelian group T ($T = \mathbf{Z}[S]$ for some set S), the complex is exact.

Also show that

$$0 \to A \otimes T \to B \otimes T \to C \otimes T \to 0$$

is a complex. Show that it is exact except that possibly the map $A \otimes T \to B \otimes T$ need not be injective. Show that the complex is exact for a free abelian group T.

Exercise 3.10. The (simplicial) *Klein bottle* K is the one corresponding to the following picture (note that in comparison to the projective plane, the direction of the right vertical edge has changed, and there is only one vertex):



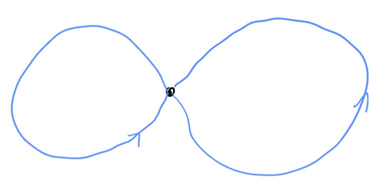
- Define this simplicial set formally. Hint: start with $\Delta^2 \sqcup \Delta^2$, corresponding to α and β, and define a number of intermediate simplicial sets by glueing certain simplices step by step.

- Spell out the normalized chain complex $N(\mathbf{Z}[K])$.

- Compute the homologies $\mathrm{H}_k(K)$.

- (Optional, bonus) Compute $\mathrm{H}_k(K, \Lambda)$ for an arbitrary ring Λ. Relate your computations to the explanation of the universal coefficient theorem made after Definition 3.16.

Exercise 3.11. Let $X := S^1 \sqcup_{\{*\}} S^1$ be two copies of the simplicial sphere, glued together at the unique 0-simplex.



Show

$$\mathrm{H}_n(\mathbf{Z}[X]) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z}^2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 3.12. Compute the homology of the (simplicial) lasso, cf. Exercise 2.2.

Exercise 3.13. The *set of path components* of a simplicial set X is defined as

$$\pi_0(X) := X_0/\sim,$$

where $\sim$ is the equivalence relation generated by the following relation $\sim'$ (for $x_0, x_1 \in X_0$): $x_0 \sim' x_1$ iff there is an edge $e \in X_1$ such that $d_k(e) = x_k$ for $k = 0, 1$.

- Show that $x_0 \sim' x_1$ iff the two maps $\Delta^0 \to X$ given by x_0, x_1 (via the Yoneda lemma, cf. Lemma 2.33) are homotopic.
- Given an example of a simplicial set X where the relation $\sim'$ is not an equivalence relation.
- Show that for $X = \mathrm{Sing}(Y)$, for a topological space Y, the relation $\sim'$ is, however, an equivalence relation (so that $\sim = \sim'$ in this case). Prove

  $$\pi_0(\mathrm{Sing}(Y)) = \pi_0(Y),$$

  where the right hand side is the set of path components (cf. Exercise 1.3).
- Show that $\mathrm{H}_0(X)$ is isomorphic to $\mathbf{Z}[\pi_0(X)]$ for any simplicial set X.
- (Optional, bonus): Show that the assignment $X \mapsto \pi_0(X)$ gives rise to a functor $\mathrm{sSet} \to \mathrm{Set}$. Show that this functor is left adjoint to the discrete-simplicial-set functor $\mathrm{disc} : \mathrm{Set} \to \mathrm{sSet}$ (Example 2.4).

Chapter 4

Singular homology

In this chapter, we finally introduce homology of topological spaces. We also prove the Eilenberg–Steenrod axioms: the dimension axiom, the additivity for homology, as well as the homotopy invariance, the Mayer–Vietoris sequence and the (essentially equivalent)excision. We use these to compute homology groups of various spaces including spheres and projective spaces. These computations are used to prove the Brouwer fixed point theorem (cf. §1.2), as well as the Borsuk–Ulam theorem and the fundamental theorem of algebra.

4.1 Definition

Definition 4.1. The n-th *singular homology* (or just *homology*) of a topological space X is defined to be

$$\mathrm{H}_n(X) := \mathrm{H}_n(\mathrm{Sing}(X))(:= \mathrm{H}_n(N(\mathbf{Z}[\mathrm{Sing} X]))).$$

More diagrammatically, $\mathrm{H}_n(X)$ is the image of X under the following composition of functors:

$$\mathrm{Top} \stackrel{\mathrm{Sing}}{\to} \mathrm{sSet} \stackrel{\mathbf{Z}[-]}{\to} \mathrm{sAb} \stackrel{N}{\to} \mathrm{Ch} \stackrel{\mathrm{H}_n}{\to} \mathrm{Ab}.$$

Being a composite of functors, H_n is itself a functor:

$$\mathrm{H}_n : \mathrm{Top} \to \mathrm{Ab}.$$

Again, slightly more generally, for a commutative ring Λ, we define homology with Λ-coefficients as

$$\mathrm{H}_n(X, \Lambda) := \mathrm{H}_n(N(\Lambda[\mathrm{Sing} X]))(\in \mathrm{Mod}_\Lambda).$$

Remark 4.2. By Exercise 4.6, the maps

$$\mathrm{H}_n(C(\mathbf{Z}[\mathrm{Sing} X])) \to \mathrm{H}_n(N(\mathbf{Z}[\mathrm{Sing} X]))$$

are isomorphisms, so that we can interchange the chain complexes C and its normalized variant N at will.

4.2 Dimension axiom and additivity

Proposition 4.3. (*Dimension axiom*) We have

$$\mathrm{H}_n(\{*\}) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Indeed, since any continuous map $\Delta^n_{\mathrm{top}} \to \{*\}$ factors over Δ^0_{top}, $\mathrm{Sing}(\{*\})$ is the discrete simplicial set associated to a point, i.e., Δ^0. Then $N(\mathbf{Z}[\Delta^0]) = \mathbf{Z}$ (in degree 0), which has the homology stated above. □

Proposition 4.4. (*Additivity*) Homology of a disjoint union of spaces can be computed as

$$\mathrm{H}_n\left(\coprod_{i \in I} X_i\right) = \bigoplus_{i \in I} \mathrm{H}_n(X_i).$$

In the proof we use the *direct sum* of two complexes C and D, which is simply given by

$$\dots \to (C \oplus D)_n := C_n \oplus D_n \stackrel{\partial^C_n \oplus \partial^D_n}{\to} C_{n-1} \oplus D_{n-1} \to \dots$$

Proof. Indeed, each of the functors in the diagram below preserves coproducts (which are disjoint unions in Top, and direct sums in the three right hand categories):

$$\mathrm{Top} \stackrel{\mathrm{Sing}}{\to} \mathrm{sSet} \stackrel{\mathbf{Z}[-]}{\to} \mathrm{sAb} \stackrel{C}{\to} \mathrm{Ch} \stackrel{\mathrm{H}_n}{\to} \mathrm{Ab}.$$

(Alternatively, N can also be used in place of C.) For Sing, this was shown in Lemma 2.30 (this used that $\Delta^n (\in \mathrm{Top})$ is connected). For the free abelian group functor $\mathbf{Z}[-] : \mathrm{Set} \to \mathrm{Ab}$, this is clear from the definition (or from the fact that it is a left adjoint). By definition, both C and N also preserves direct sums, i.e.,

$$C(\bigoplus A_i) = \bigoplus C(A_i)$$

for a family of simplicial abelian groups $A_i \in \mathrm{sAb}$, and likewise for the normalized chain complex N.

Next, the cycle and boundary complex functors preserve direct sums, i.e., for a family of chain complexes C_i the direct sum $\bigoplus_i C_i$ is given in degrees n and $n-1$ by

$$\bigoplus_i (C_i)_n \xrightarrow{\partial = \bigoplus_i \partial_{C_i}} \bigoplus_i (C_i)_{n-1}.$$

Thus, being a cycle, resp. a boundary in this complex means that each component (for all i) is a cycle, resp. a boundary:

$$Z_n(\bigoplus_i C_i) = \bigoplus_i Z_n(C_i), B_n(\bigoplus_i C_i) = \bigoplus_i B_n(C_i).$$

Using finally that direct sums (of abelian groups) commute with quotients (i.e., for a family of subgroups $V_i \subset W_i$, we have $\bigoplus W_i / \bigoplus V_i = \bigoplus (W_i/V_i)$), we are done since $\mathrm{H}_n = Z_n/B_n$. □

Homology in degree 0 is easy to compute. Recall that the set $\pi_0(X)$ of path components is defined as

$$\pi_0(X) := X/\sim,$$

where $x \sim y$ iff there is a continuous map $\Delta^1 \to X$ whose endpoints are x and y, respectively.

Lemma 4.5. For a topological space X,

$$\mathrm{H}_0(X) = \mathbf{Z}[\pi_0(X)]$$

is the free abelian group on the set of path-components of X.

Proof. This follows from Exercise 3.13:

$$\mathrm{H}_0(X) = \mathrm{H}_0(\mathrm{Sing}(X)) = \mathbf{Z}[\pi_0(\mathrm{Sing} X)] = \mathbf{Z}[\pi_0(X)].$$

Remark 4.6. Let X be a topological space, and $x \in X$. There is a canonical map from the *fundamental group* (with base-point x) to the first homology group:

$$\pi_1(X, x) \to \mathrm{H}_1(X)$$

defined by sending a *loop*, i.e., a continuous map $\sigma : \Delta^1 \to X$ with $\sigma((0,1)) = \sigma((1,0)) = x$), to σ. Note that $\sigma \in \mathrm{Sing}(X)_1 \subset \mathbf{Z}[\mathrm{Sing}(X)_1]$ is a cycle, since

$$d(\sigma) = d_0(\sigma) - d_1(\sigma) = \sigma((0,1)) - \sigma((1,0)) = x - x = 0.$$

Thus, $[\sigma]$ is indeed an element in $\mathrm{H}_1(X)$. One checks (!)that for another loop τ that is homotopic to σ (where the homotopy is relative to the base point x) $\sigma - \tau$ is a boundary in $C(\mathbf{Z}[\mathrm{Sing} X])$, so the map above is well-defined. One also checks that it is in fact a group homomorphism.

The so-called *Hurewicz theorem* asserts that the above map, for X being connected, the above map induces an isomorphism

$$(\pi_1(X,x))_{\mathrm{ab}} \xrightarrow{\cong} \mathrm{H}_1(X)$$

between the *abelianization* of π_1 and the homology group. See, e.g., [Rot88, Theorem 4.29] or [GJ09, Corollary III.3.6] for an exposition on the level of appropriate simplicial sets, called *Kan complexes*.

For example, the fundamental group of $\mathbf{R}^2 \backslash \{p_1, \dots, p_n\}$ can be shown (using the *Seifert–van Kampen theorem*) to be the free group on n generators (namely, loops winding around the points p_k once, but not around the others), while

$$\mathrm{H}_1(\mathbf{R}^2 \backslash \{p_1, \dots, p_n\}) = \mathbf{Z}^n.$$

(That computation requires the Mayer–Vietoris sequence below. See also Outlook 4.29 for further allusions to the similarity between the Seifert–van Kampen theorem and the Mayer–Vietoris sequence.)

Outlook 4.7. Lemma 4.5 indicates that singular homology is well-adapted to topological spaces which have enough (continuous) maps $\Delta^1 \to X$. Not all spaces are of this form, such as

- the *topologists' sine curve*, $T := \{(x, \sin(x^{-1})) | x \in (0,1]\} \cup \{(0,0)\} (\subset \mathbf{R}^2)$ which is a connected, but not path-connected topological space (see, e.g., [Hat, §2]),
- the spectrum $\mathrm{Spec}\, R$ of a commutative ring (for example $R = \mathbf{Z}$), equipped with its *Zariski topology*.

For such more general spaces, it is still possible to glean meaningful (co)homological information using so-called *sheaf cohomology*.

4.3 Homotopy

After the dimension axiom and the behavior of homology with respect to disjoint unions, the next easiest property of homology is the homotopy axiom.

Proposition 4.8. (*Homotopy axiom* or *Homotopy invariance* of homology) Homotopic maps induce the same maps on homology. More formally, let $f, g : X \to Y$ be continuous maps that are homotopic. Then the induced (chain) maps

$$N(f), N(g) : N(X) \to N(Y)$$

are homotopic, so that

$$\mathrm{H}_n(f), \mathrm{H}_n(g) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

are the *same* maps.

Corollary 4.9. If a continuous map $f : X \to Y$ is a continuous homotopy equivalence, then

$$\mathrm{H}_n(f) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$$

is an isomorphism.

Proof. (of Proposition 4.8) The point is that every functor in

$$\mathrm{Top} \overset{\mathrm{Sing}}{\to} \mathrm{sSet} \overset{N(\mathbf{Z}[-])}{\to} \mathrm{Ch} \overset{\mathrm{H}_n}{\to} \mathrm{Ab},$$

plays well with homotopies. Let h be a (continuous) homotopy between f and g. Then:

- $\mathrm{Sing}(h)$ gives rise to a simplicial homotopy h' between $f' := \mathrm{Sing}(f)$ and $g' := \mathrm{Sing}(g)$ (Proposition 2.39),
- $N(h')$ gives rise, via the Eilenberg–Zilber map ∇, to a homotopy between $N(f')$ and $N(g')$ (Proposition 3.41),
- chain homotopic maps give rise to the same maps after applying H_n (Lemma 3.24). □

Example 4.10. Let $X \subset \mathbf{R}^n$ be a non-empty convex subset (with the subspace topology). Then the inclusion $i : \{x_0\} \to X$ of any point is a (continuous) homotopy equivalence, for $p : X \to \{x_0\}$ satisfies $p \circ i = \mathrm{id}$ and $i \circ p$ is homotopic to id_X via

$$h : \Delta^1_{\mathrm{Top}} \times X \to X, ((t_0, t_1), x) \mapsto t_0 x + t_1 x_0.$$

(This map is well-defined since X is convex.)

Thus, the maps

$$\mathrm{H}_n(X) \underset{\mathrm{H}_n(i)}{\overset{\mathrm{H}_n(p)}{\rightleftarrows}} \mathrm{H}_n(\{*\})$$

are isomorphisms, so that Proposition 4.3 gives

$$\mathrm{H}_n(X) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 4.11. Recall that a subspace $A \subset X$ of a topological space X is a *deformation retract*, if there is a map $h : X \times [0,1] \to X$ such that $h_0 := h|_{X \times 0}$ is the identity, h_1 takes values in $A (\subset X)$ and $h|_{A \times [0,1]} = \mathrm{id}_A$. For example, the inclusion of S^1 into a Möbius strip is a deformation retract.

Then the inclusion $i : A \subset X$ induces isomorphisms

$$\mathrm{H}_n(i) : \mathrm{H}_n(A) \to \mathrm{H}_n(X).$$

Indeed, the (continuous!) map $h_1 : X \to A$ is such that $h_1 \circ i = \mathrm{id}_A$, while $i \circ h_1$ is homotopic (via h) to $i \circ h_0 = \mathrm{id}_A$.

The homotopy axiom can also be recast using a categorical language, by using the category HoTop (called *topological spaces up to homotopy*) whose objects are topological spaces and

$$\mathrm{Hom}_{\mathrm{HoTop}}(X, Y) := \mathrm{Hom}_{\mathrm{Top}}(X, Y)/\sim,$$

where $\sim$ is the homotopy (equivalence!) relation, cf. Remark 2.38. One checks that this is indeed a category (the point being that if $f, g : X \to Y$ satisfy $f \sim g$ then $f \circ e \sim g \circ e$ and $e \circ f \sim e \circ g$ for appropriate continuous maps e). Similar definitions yield categories HosSet and HoCh (cf. Exercise 3.2) for the latter. The homotopy axiom (and its proof!) can then be restated using the following diagram

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, "\mathrm{Sing}"] \arrow[d] & \mathrm{sSet} \arrow[r, "N(\mathbf{Z}[-])"] \arrow[d] & \mathrm{Ch} \arrow[r, "\mathrm{H}_n"] \arrow[d] & \mathrm{Ab} \arrow[d, equal] \\
\mathrm{HoTop} \arrow[r, dotted, "2.39"'] & \mathrm{HosSet} \arrow[r, dotted, "3.41"'] & \mathrm{HoCh} \arrow[r, dotted, dash, "3.24"'] & \mathrm{Ab}.
\end{tikzcd}$$

4.4 Mayer–Vietoris sequences

So far, we have been able to compute the homology of topological spaces X such that X is isomorphic in HoTop to a point. The remaining key property of homology will allow us to drop that restriction. The basic idea of Mayer–Vietoris sequences and excision is to break the computation of homology of some space X into the homology of smaller, hopefully more easily understood, subspaces of X.

4.4.1 Preliminaries from homological algebra

Definition 4.12. A *short exact sequence of abelian groups* is an exact complex

$$\ldots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \ldots$$

which we will abbreviate as

$$0 \to A \to B \to C \to 0.$$

(Concretely, $g \circ f = 0$, f is injective, $\ker g = \operatorname{im} f$ and g is surjective.) More generally, the same definition applies to general abelian categories such as Mod_Λ or $\mathrm{Ch}(\mathrm{Mod}_\Lambda)$ for any ring Λ, instead of abelian groups. In particular, a *short exact sequence of chain complexes* is a sequence of chain maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

whose evaluation in each degree $n \in \mathbf{Z}$ gives an exact sequence in the above sense.

The following lemma is immensely useful in practice. We will use it to obtain the highly useful Mayer–Vietoris sequence.

Lemma 4.13. (*Snake lemma* or "Short exact sequences of chain complexes give rise to long exact sequences of homology groups.") Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of chain complexes. Then there is a long exact sequence of homology groups

$$\ldots \to \mathrm{H}_n(A) \xrightarrow{\mathrm{H}_n(f)} \mathrm{H}_n(B) \xrightarrow{\mathrm{H}_n(g)} \mathrm{H}_n(C) \xrightarrow{d} \mathrm{H}_{n-1}(A) \to \ldots$$

where the so-called *connecting homomorphism* is a group homomorphism whose idea is the following "definition":

$$d(c) = \text{“}f^{-1}\partial_B g^{-1}(c)\text{”}.$$

Proof. We only make precise the definition of d, referring to [Wei94, Theorem 1.3.1] for a complete proof.

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "\partial"] & B_n \arrow[r, "g_n"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] & B_{n-1} \arrow[r, "g_{n-1}"] & C_{n-1} \arrow[r] & 0.
\end{tikzcd}$$

$$\begin{tikzcd}
& b \arrow[r, mapsto, "g_n"] \arrow[d, "\partial"] & c \arrow[r] \arrow[d, mapsto, "\partial"] & 0 \\
a \arrow[r, mapsto, "f_{n-1}"] \arrow[d, mapsto, "\partial"] & \partial b \arrow[r, "g_{n-1}"] \arrow[d, mapsto, "\partial"] & 0 & \\
0 \arrow[r, mapsto, "f_{n-2}"] & \partial\partial b = 0. & &
\end{tikzcd}$$

We define a map $\tilde{d} : Z_n(C) \to \mathrm{H}_n(A)$: let $c \in Z_n(C)$. We can choose some $b \in B_n$ with $g(b) = c$. Then $g(\partial b) = \partial g(b) = \partial c = 0$, so that there is a unique (by exactness of the bottom sequence) $a \in A_n$ with $f(a_n) = \partial b$. Define $\tilde{d}(c) := a_n \in A_n$. We have $\partial\partial b = 0$, so that $\tilde{d}(c) \in Z_n(A)$. The element $\tilde{d}(c)$ so defined depends on the choice of b, but as an element in $\mathrm{H}_n(A) = Z_n(A)/B_n(A)$, this is independent of the choice: any other b' with this property satisfies $b - b' \in A_n$, so that $\partial b - \partial b' \in B_n(A)$ (more precisely, there is an element in A_{n+1} whose image under f_{n+1} is $\partial b - \partial b'$). The map $\tilde{d}$ factors over $\mathrm{H}_n(C)$: if $c = \partial c' \in B_n(C)$, we can choose $b' \mapsto c'$ and then $b := \partial b' \mapsto c = \partial c'$. Then $\tilde{d}(c) = \partial b = \partial\partial b' = 0$. $\square$

Example 4.14. Let C be a complex of free (or just torsion-free) abelian groups and $\ell \in \mathbf{Z}$ an integer. Write C/ℓ for the complex $\dots C_n/\ell \xrightarrow{\partial} C_{n-1}/\ell \dots$. Then there is a short exact sequence

$$0 \to C \xrightarrow{\ell} C \to C/\ell$$

and hence a long exact sequence

$$\dots \to \mathrm{H}_n C \xrightarrow{\ell} \mathrm{H}_n C \to \mathrm{H}_n(C/\ell) \to \mathrm{H}_{n-1}(C) \to \dots.$$

This long exact sequence can be broken up (Exercise 4.9) into short exact sequences

$$0 \to \mathrm{H}_n(C)/\ell \to \mathrm{H}_n(C/\ell) \to (\mathrm{H}_{n-1}(C))_\ell \to 0,$$

where the right hand term denotes the ℓ-*torsion* part of the group ($M_\ell := \{m \in M, \ell m = 0\}$).

4.4.2 Construction of Mayer–Vietoris sequences

In this section, let

$$\mathcal{U} = \{U_i\}_{i \in I}$$

be a collection of (not necessarily open) subspaces $U_i \subset X$ of some topological space X.

We define a sub-simplicial set $\mathrm{Sing}^{\mathcal{U}}(X) \subset \mathrm{Sing}(X)$ to consist of those n-simplices $f : \Delta^n_{\mathrm{top}} \to X$ such that $f(\Delta^n_{\mathrm{top}}) \subset U_i$ for some i. This is(!)indeed a simplicial set.

Example 4.15. If $X = U \cup V$ is the union of two subspaces, then we have a commutative diagram of simplicial sets

$$\begin{tikzcd} \mathrm{Sing}(U \cap V) \arrow[r] \arrow[d] & \mathrm{Sing}(U) \arrow[d] \\ \mathrm{Sing}(V) \arrow[r] & \mathrm{Sing}^{\mathcal{U}}(X). \end{tikzcd} \tag{4.16}$$

- It is a pullback square: a continuous map $\Delta^n \xrightarrow{u} U$ and another $\Delta^n \xrightarrow{v} V$ whose composition to X is the same map $\Delta^n \to X$ is the same as a continuous map $\Delta^n \to U \cap V$.
- Very importantly, it is also a pushout, by the very definition of $\mathrm{Sing}^{\mathcal{U}}(X)$.

- Unless $U \cap V = \varnothing$, $\mathrm{Sing}(X)$ is *not* the pushout of the above diagram: a map $\Delta^n \to X$ need not factor over U or V, so that

$$\mathrm{Sing}(U)_n \sqcup \mathrm{Sing}(V)_n \to \mathrm{Sing}(X)_n$$

  is not surjective. See, however, Outlook 4.29, for more positive remarks.

Lemma 4.17. Let

$$\begin{tikzcd} U \cap V \arrow[r, "j'"] \arrow[d, "i'"] & U \arrow[d, "i"] \\ V \arrow[r, "j"] & X \end{tikzcd}$$

be some subspaces of a topological space X. There is a short exact sequence of complexes

$$0 \to C(U \cap V) \stackrel{j'_* + i'_*}{\to} C(U) \oplus C(V) \stackrel{i_* - j_*}{\to} C^{\mathcal{U}}(X) \to 0.$$

Here we abbreviate $C(U) := C(\mathbf{Z}[\mathrm{Sing}(U)])$, $C^{\mathcal{U}}(X) := C(\mathbf{Z}[\mathrm{Sing}^{\mathcal{U}}(X)])$ etc.

Notation 4.18. Above, and also in the sequel, we write

$$f_*$$

for the evaluation of some functor (which is often implicit) $\mathrm{Top} \to C$ on f. For example, for $i : U \to X$, $i_* := C(i) : C(U) \to C(X)$, and likewise we would write $i_* := \mathrm{H}_k(i) : \mathrm{H}_k(U) \to \mathrm{H}_k(X)$.

Proof. In each simplicial degree, the diagram (4.16) gives a diagram of sets, that is again both a pullback and pushout:

$$\begin{tikzcd} \mathrm{Sing}_n(U \cap V) \arrow[r] \arrow[d] & \mathrm{Sing}_n(U) \arrow[d] \\ \mathrm{Sing}(V)_n \arrow[r] & \mathrm{Sing}^{\mathcal{U}}_n(X). \end{tikzcd}$$

By Exercise 4.4, taking the free abelian groups on these sets gives an exact sequence which is the n-the degree of the claimed exact sequence. $\square$

A priori, the simplicial set $\mathrm{Sing}^{\mathcal{U}}(X)$ looks unwieldy, but the following key theorem relates it back to something we know (and care about!). It is sometimes referred to as *locality*, with the idea that it says that the homology of X is completely determined by how X looks locally.

Theorem 4.19. Suppose

$$X = \bigcup_i U_i^\circ,$$

i.e., X is covered by the *interiors* of the U_i. (Recall the interior of $A \subset X$ is the largest open subset $U \subset X$ that is still contained in A. Thus, if the U_i are open, the condition just means $X = \bigcup_i U_i$.) Then the inclusion

$$i : \mathrm{Sing}^{\mathcal{U}}(X) \to \mathrm{Sing}(X)$$

induces a chain homotopy equivalence

$$i : C^{\mathcal{U}}(X) := C(\mathbf{Z}[\mathrm{Sing}^{\mathcal{U}}(X)]) \to C(X) := C(\mathbf{Z}[\mathrm{Sing}(X)]).$$

Therefore, the homologies of these two complexes are isomorphic.

Corollary 4.20. (*Mayer–Vietoris sequence*) Let

$$\begin{tikzcd} U \cap V \arrow[r, "j'"] \arrow[d, "i'"] & U \arrow[d, "i"] \\ V \arrow[r, "j"] & X \end{tikzcd}$$

be some subspaces of a topological space X such that

$$X = U^\circ \cup V^\circ.$$

Then there is a long exact sequence of homology groups

$$\ldots \to \mathrm{H}_n(U \cap V) \xrightarrow{j'_* + i'_*} \mathrm{H}_n(U) \oplus \mathrm{H}_n(V) \xrightarrow{i_* - j_*} \mathrm{H}_n(X) \xrightarrow{d} \mathrm{H}_{n-1}(U \cap V) \to \ldots.$$

Proof. This follows from Lemma 4.17, Theorem 4.19, and Lemma 4.13. □

4.4.3 Homology of spheres

Proposition 4.21. The homology of the (topological) k-sphere

$$S^k = \{(x_0, \ldots, x_k), \sum_i x_i^2 = 1\}$$

is given for $k \neq 0$ by

$$\mathrm{H}_n(S^k) = \begin{cases} \mathbf{Z} & n = 0 \\ \mathbf{Z} & n = k \\ 0 & \text{otherwise} \end{cases}$$

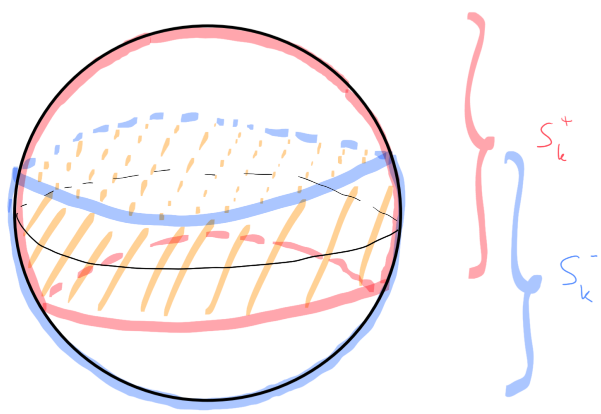
and

$$\mathrm{H}_n(S^0) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $0 < \epsilon < \frac{1}{2}$. We consider the covering

$$S^k = S^k_+ \cup S^k_-$$

where S^k_+ consists of the points with $x_k > -\epsilon$ and S^k_- of those with $x_k < \epsilon$.



Both $S^k_\pm$ are homeomorphic to the k-dimensional disk D^k, so that $\mathrm{H}_n(S^k_\pm) = \mathbf{Z}$ in degree $n = 0$ and 0 else. The intersection $S^k_+ \cap S^k_-$ is homeomorphic to $S^{k-1} \times (-\epsilon, \epsilon)$. By the homotopy axiom, its homology is therefore isomorphic to the one of S^{k-1}. The Mayer–Vietoris sequence then reads

$$\ldots \to \mathrm{H}_n(S^{k-1}) \xrightarrow{j'_* + i'_*} \underbrace{\mathrm{H}_n(D^k) \oplus \mathrm{H}_n(D^k)}_{=0 \text{ for } n \neq 0} \xrightarrow{i_* - j_*} \mathrm{H}_n(S^k) \xrightarrow{d} \mathrm{H}_{n-1}(S^{k-1}) \to \underbrace{\mathrm{H}_{n-1}(D^k) \oplus \mathrm{H}_{n-1}}_{=0 \text{ for } n \neq 1}$$

This shows that

$$d : \mathrm{H}_n(S^k) \to \mathrm{H}_{n-1}(S^{k-1})$$

is an isomorphism unless $n = 0$ or 1. For $n = 0$, we already know $\mathrm{H}_0(S^n) = \mathbf{Z}$ for $n > 0$ and $\mathbf{Z} \oplus \mathbf{Z}$ for $n = 0$ (Lemma 4.5). For $n = 1$, we get a sequence

$$0 \to \mathrm{H}_1(S^k) \to \mathrm{H}_0(S^{k-1}) \xrightarrow{j'_* \oplus i'_*} \mathrm{H}_0(S^+_k) \oplus \mathrm{H}_0(S^k_-).$$

For $k = 1$, the right hand map identifies with

$$\mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbf{Z} \oplus \mathbf{Z},$$

with kernel $\mathrm{H}_1(S^1) = \{(x, -x)\} \cong \mathbf{Z}$. For $k > 1$, the right hand map identifies with

$$\mathbf{Z} \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \mathbf{Z} \oplus \mathbf{Z},$$

which is injective, so that $\mathrm{H}_1(S^k) = 0$. □

Pending the proof of the excision property, we at this point have proved the Brouwer fixed point theorem, as well as the topological invariance of dimension in §1.2.

4.4.4 Proof

We now prove Theorem 4.19. The proof is a combination of two ideas:

(1) We devise a way to break simplices into smaller pieces (in a way that is a chain homotopy equivalence), by using the barycentric subdivision (Lemma 4.24). For technical purposes, we do this construction just on Δ^n.

(2) Using the Lebesgue covering lemma, we iterate the construction (now performed on our space X) for each simplex individually, so that the resulting simplices are small enough to fit into one of the U_i.

This will suffice to construct a map $\tilde{S}$ that exhibits the inclusion i to be a chain homotopy equivalence:

$$\begin{tikzcd} C^{\mathcal{U}}(X) \arrow[r, hook, "i"'] & C(X) \arrow[l, bend right, dotted, "\tilde{S}"'] \end{tikzcd}.$$

For a convex subspace $Y \subset \mathbf{R}^n$ (such as $Y = \Delta^k$), we define a subcomplex $C'(Y) \subset C(Y)$ given in degree n by the *affine-linear* maps $\sigma : \Delta^n \to Y$. Recall that this means that the map σ is given by

$$\sigma(t_0, \dots, t_n) = \sum_{k=0}^{n} t_k \sigma(e_k),$$

with $e_k \in \mathbf{R}^{n+1}$ being the k-th standard basis vector.

Then $C'(Y)$ is indeed a subcomplex, since the faces of such an affine-linear simplex are again affine-linear. There is a group isomorphism

$$\mathbf{Z}[Y^{n+1}] \xrightarrow{\cong} C'(Y)_n \tag{4.22}$$

that maps a tuple $(y_0, \dots, y_n)$ to the unique affine-linear map $\Delta^n \to Y$ sending the standard basis vector $e_i \mapsto y_i$. We denote that simplex in Y by $[y_0, \dots, y_n]$.

A point $y \in Y$ gives rise to a homomorphism

$$b_y : C'(Y)_n \to C'(Y)_{n+1}, [y_i] \mapsto [y, y_0, \dots, y_n].$$

The map b_y can be thought of replacing an n-simplex by a cone whose tipping point is y and whose base is that simplex. The map is *not* a chain morphism, but instead we have

$$\partial b_y = \mathrm{id} - b_y \partial. \tag{4.23}$$

Lemma 4.24. There is a subdivision chain map

$$S : C'(Y) \to C'(Y),$$

defined inductively as the identity in chain degree 0, and for $\lambda : \Delta^n \to Y$, as

$$S(\lambda) := b_\lambda(S \partial \lambda),$$

where b_λ is the map b associated to the point $\lambda(\sum_{k=0}^{n} \frac{e_k}{n+1}) \in Y$.

This chain map is homotopic to the identity.

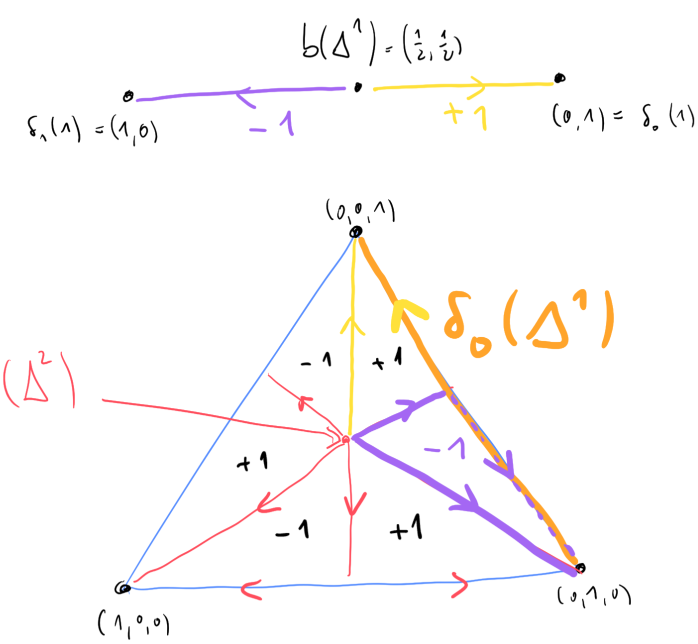
Example 4.25. We unwind this definition for $Y = \Delta^2$, and 0, 1, and 2-simplices. For a 0-simplex y, we have $S(y) = y$. Now, we compute $S(\lambda)$, where $\lambda := \delta_0 : \Delta^1 \to \Delta^2$ is the map defined in (2.16), i.e., $\delta_0(t_0, t_1) = (0, t_0, t_1)$. We have

$$\partial \lambda = d_0(\lambda) - d_1(\lambda) = \lambda \circ \delta_0 - \lambda \circ \delta_1$$

where now $\delta_k : \Delta^0 \to \Delta^1$ are the maps $\delta_0 : t(=1) \mapsto (0,t)$, $\delta_1 : t \mapsto (t,0)$. We have to consider the barycentric subdivision map $b_\lambda = b_{\lambda(\frac{1}{2},\frac{1}{2})} = b_{(0,\frac{1}{2},\frac{1}{2})}$. It sends the point $d_0(\lambda) = (0,0,1)$ to the 1-simplex $[(0,\frac{1}{2},\frac{1}{2}),(0,0,1)]$ etc., so that

$$S(\delta_0) = [(0,\frac{1}{2},\frac{1}{2}),(0,0,1)] - [(0,\frac{1}{2},\frac{1}{2}),(0,1,0)].$$

Note that the first summand is a 1-simplex whose endpoint (i.e., applying d_0) is $(0,0,1)$ and whose beginning point (i.e., d_1 of it) is $(0,\frac{1}{2},\frac{1}{2})$. and $S(\mathrm{id}_{\Delta^2})$ is a formal linear combination of six triangles inside Δ^2, with signs as shown:



Proof. We have to prove $S\partial = \partial S$. This is clear in degree 0. In higher degrees, we argue inductively

$$\begin{aligned}
\partial S\lambda &= \partial\left(b_\lambda(S\partial\lambda)\right) \\
&= S\partial\lambda - b_\lambda(\partial S\partial\lambda) && \text{(by (4.23))} \\
&= S\partial\lambda - b_\lambda(S\partial^2\lambda) && \text{(by induction)} \\
&= S\partial\lambda && \text{($C'(Y)$ is a chain complex).}
\end{aligned}$$

In order to define a homotopy between S and $\mathrm{id}_{C'(Y)}$, it is notationally convenient to enlarge this complex by replacing the low degrees by

$$\ldots \to C'_0(Y) = \bigoplus_{y \in Y} \mathbf{Z} \xrightarrow{\sum n_y y \mapsto \sum n_y} C'_{-1} := \mathbf{Z}.$$

The isomorphism (4.22) persists also for $n = -1$ then. We also put $S := \mathrm{id}$ in degree -1.

We define a homotopy between S and $\mathrm{id}_{C'(Y)}$:

$$\begin{tikzcd}
\ldots \arrow[r] & C'_2(Y) \arrow[r, "\partial"] \arrow[d, "S"] & C'_1(Y) \arrow[r, "\partial"] \arrow[d, "S"] \arrow[dl, dotted, "h_1"'] & C'_0(Y) \arrow[r, "\partial"] \arrow[d, "S=\mathrm{id}"] \arrow[dl, dotted, "h_0"'] & C'_{-1}(Y) \arrow[r] \arrow[d, "S=\mathrm{id}"] \arrow[dl, dotted, "h_{-1}"'] & 0 \\
\ldots \arrow[r] & C'_2(Y) \arrow[r, "\partial"] & C'_1(Y) \arrow[r, "\partial"] & C'_0(Y) \arrow[r, "\partial"] & C'_{-1}(Y) \arrow[r] & 0.
\end{tikzcd}$$

We define $h_{-1} := 0$ and in degrees $\geqslant 0$ as

$$h\lambda := b_\lambda(\lambda - h\partial\lambda).$$

The claim is now

$$\partial h + h\partial = \mathrm{id} - S.$$

This is clear in degree -1. In higher degrees:

$$\begin{aligned}
\partial h\lambda &= \partial(b_\lambda(\lambda - h\partial\lambda)) \\
&= \lambda - h\partial\lambda - b_\lambda(\partial(\lambda - h\partial\lambda)) && by (4.23) \\
&= \lambda - h\partial\lambda - b_\lambda(S\partial\lambda + h\partial^2\lambda) && \text{by induction} \\
&= \lambda - h\partial\lambda - S\lambda && \text{by definition of } S.
\end{aligned}$$

At this point we can discard the (-1)-st degree of $C'(Y)$; we still have a homotopy S as stated, since $h_{-1} = 0$. □

Lemma 4.26. The subdivision maps S constructed above give rise to a chain map

$$\tilde{S} : C(X) \to C^{\mathcal{U}}(X)$$

such that the two composites with the inclusion $i : C^{\mathcal{U}}(X) \subset C(X)$ are homotopic to the identities.

Proof. The proof is based on the idea that subdividing a simplex $\sigma : \Delta^n \to X$ often enough, say m times, it will be a linear combination of simplices that each lies in some U_i. The proof is more tricky though since the number m will depend on the simplex σ.

We first transport the idea of taking barycentric subdivisions on (the convex space) Δ^n to X: the maps $S : C'(\Delta^n) \to C'(\Delta^n)$ give rise to a map

$$S : C(X) \to C(X), \sigma \mapsto \sigma_*(S\mathrm{id}_{\Delta^n}).$$

This is a chain map:

$$\begin{aligned}
\partial S\sigma &:= \partial(\sigma_* S\mathrm{id}) \\
&= \sigma_* \partial S\mathrm{id} \\
&= \sigma_* S \partial \mathrm{id} \\
&= \sigma_* S \left(\sum_{k=0}^{n} (-1)^k \delta_k \right), \text{ with } \delta_k : \Delta^{n-1} \to \Delta^n \\
&= \sum_k (-1)^k S(\sigma \circ \delta_k) \\
&= S(\partial \sigma).
\end{aligned}$$

By a similar computation, this chain map S is homotopic to the identity via $h_n(\sigma) = \sigma_* h(\mathrm{id}_{\Delta^n})$.

The m-fold iterate S^m is chain homotopic to S^{m-1} via $h \circ S^{m-1}$, so that $h^{(m)} := \sum_{k=0}^{m-1} hS^k$ is a chain homotopy between $\mathrm{id} = S^0$ and S^m.

For any simplex σ in $\mathrm{Sing}(X)$, there is some $m(\sigma) \gg 0$ such that $S^{m(\sigma)}(\sigma) \in C^{\mathcal{U}}(X)$. Indeed,

$$\Delta^n = \sigma^{-1}(X) = \bigcup_i \sigma^{-1}(U_i^\circ)$$

is an open covering of a compact metric space, so by the *Lebesgue covering lemma* (see, e.g., [Mun00, Lemma 27.5]) there is some $\epsilon > 0$ such that for each $x \in \Delta^n$, the open ball $B(x, \epsilon)$ is contained in one of the $\sigma^{-1}(U_i)$, i.e., $f(B(x, \epsilon)) \subset U_i$.

Let $r := \mathrm{diam}(\Delta^n)$ be the diameter of Δ^n. The diameter of the simplices appearing in the barycentric subdivision, i.e., in $S(\mathrm{id}_{\Delta^n})$, is bounded by $\frac{n}{n+1} r < r$. To see this, it suffices to see that for $b := b(v_0, \dots, v_n)$ we have $d(b, v_i) < \frac{n}{n+1} r$. Indeed, if b' denotes the barycenter of $[v_0, \dots, \widehat{v}_i, \dots, v_n]$, then $b = \frac{1}{n+1} v_i + \frac{n}{n+1} b'$, so that $d(b, v_i) = \frac{n}{n+1} d(b', v_i) \leqslant \frac{n}{n+1} \mathrm{diam}(\Delta^{n-1})$.

Therefore for $m \gg 0$, the chain $S^m(\mathrm{id}_{\Delta^n})$ consists of simplices which have diameter $< \epsilon$, and therefore each lie in some U_i. Therefore, again for $m \gg 0$ the chain $S^m(\sigma)$ (which consists of the simplices in the m-fold iteration of the barycentric subdivision) lies in $C^{\mathcal{U}}(X)_n$.

Let us pick, for each σ individually, the smallest such index, which we denote by $m(\sigma)$. We contend that the maps

$$\tilde{h} : C(X)_n \to C(X)_{n+1}, \sigma \mapsto h^{(m(\sigma))}(\sigma)$$

form the required homotopy between id and a chain map $\tilde{S}$ that we we define below.

Indeed, starting from the homotopy relation

$$\partial h^{(m\sigma)}\sigma + h^{(m(\sigma))}\partial\sigma = \sigma - S^{m(\sigma)}\sigma,$$

we get, using $\partial^2 = 0$:

$$\partial\tilde{h}\sigma + \tilde{h}\partial\sigma = \sigma - \left(\underbrace{S^{m(\sigma)}\sigma + h^{(m(\sigma))}(\partial\sigma) - \tilde{h}(\partial\sigma)}_{=:\tilde{S}(\sigma)}\right).$$

Taking this as the definition of $\tilde{S}$, we get an equation

$$\partial\tilde{h} + \tilde{h}\partial = \mathrm{id} - \tilde{S}. \tag{4.27}$$

We now check $\tilde{S}(\sigma) \in C_n^{\mathcal{U}}(X)$. This is clear for $S^{(m(\sigma)}(\sigma)$. As for $(h^{(m(\sigma)} - \tilde{h})(\partial\sigma)$, we note $\partial\sigma$ is the alternating sum of the faces of σ. Let τ be one of these faces. Then $m(\tau) \leqslant m(\sigma)$. Thus $h^{(m(\sigma))}(\tau) - \tilde{h}(\tau) = \sum_{k=m(\tau)+1}^{m(\sigma)} hS^k(\tau)$ arises by applying h to a some chain each of whose summands lies in some U_i, i.e., in total it lies in $C^{\mathcal{U}}(X)_n$. Since h preserves the property of simplices being contained in some U_i, this shows $\tilde{S}(\sigma) \in C_n^{\mathcal{U}}(X)$.

We thus get a map

$$\tilde{S} : C_n(X) \to C_n^{\mathcal{U}}(X).$$

It is in fact a chain map since, by (4.27) (and $\partial^2 = 0$):

$$\partial\tilde{S} = \partial - \partial\tilde{h}\partial = \tilde{S}\partial.$$

Thus, $i \circ \tilde{S}$ is homotopic to the identity on $C(X)$. Conversely, $\tilde{S} \circ i = \mathrm{id}$, since for $\sigma \in C^{\mathcal{U}}(X)$, we have $m(\sigma) = 0$, so that $\tilde{h}(\sigma) = 0$. $\square$

This marks the end of the proof of Theorem 4.19.

Remark 4.28. The proofs of Proposition 4.8 and Theorem 4.19 (and therefore all their corollaries) hold without any changes for arbitrary coefficient rings Λ. This can be seen either by inspecting the proofs or by noting that these proofs eventually rest on certain homotopies, and using that a chain homotopy h between two chain maps $f, g : C \to D$ gives rise to a chain homotopy $h \otimes \mathrm{id}_\Lambda : N(\Delta^1) \otimes (C \otimes \Lambda) \to D \otimes \Lambda$ between $f \otimes \mathrm{id}_\Lambda, g \otimes \mathrm{id}_\Lambda : C \otimes \Lambda \to D \otimes \Lambda$.

Outlook 4.29. In the course of the proof, we have made extensive use of the barycentric subdivision and the ability to add and subtract elements in the chain complexes $C^{\mathcal{U}}(X)$. With more homotopy-theoretic prerequisites, one can prove that for $X = U^\circ \cup V^\circ$ the square

$$\begin{tikzcd}
\mathrm{Sing}(U \cap V) \arrow[r] \arrow[d] & \mathrm{Sing}(U) \arrow[d] \\
\mathrm{Sing}(V) \arrow[r] & \mathrm{Sing}(X).
\end{tikzcd}$$

is a so-called *homotopy pushout square* of simplicial sets [Lur, Tag 012C], which in the present case means that the inclusion

$$\mathrm{Sing}^{\mathcal{U}}(X) \subset \mathrm{Sing}(X)$$

is a so-called *weak equivalence*, i.e., it is a map of simplicial sets which induces an isomorphism on all homotopy groups

$$\pi_n(\mathrm{Sing}^{\mathcal{U}}(X)) \cong \pi_n(\mathrm{Sing}(X)).$$

This statement can be shown to imply the parallel statement that $C^{\mathcal{U}}(X) \to C(X)$ is a quasi-isomorphism of chain complexes. However, the former is in fact a finer statement: it can be used to prove the *Seifert–van Kampen theorem* [Lur, Tag 012M] which expresses the fundamental group(oid) of X in terms of the ones of U, V and $U \cap V$. These (possibly) non-abelian group(oids) are not accessible with homological methods.

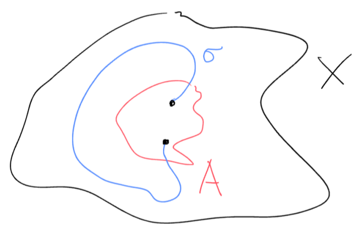
4.5 Excision

The Mayer–Vietoris sequences proved above can be equivalently recast in a form that relates the homology of a space X, a subspace $A \subset X$ and, in good cases, the quotient X/A.

Definition 4.30. Let $A \subset X$ be a subspace of a topological space. Then the *relative homology* of X with respect to A is defined to be

$$\mathrm{H}_n(X, A) := \mathrm{H}_n(C(X)/C(A)).$$

The following is a simplex $\sigma \in \mathrm{Sing}_1(X)$ that is not a cycle in $C(X)$, but is a cycle in $C(X)/C(A)$:



From the definition and the snake lemma (Lemma 4.13), we get long exact sequences

$$\ldots \to \mathrm{H}_{n+1}(X, A) \to \mathrm{H}_n(A) \to \mathrm{H}_n(X) \to \mathrm{H}_n(X, A) \to \mathrm{H}_{n-1}(A), \tag{4.31}$$

so that the relative homology measures the difference (in homology) between X and A. For example, if (for some n), the relative homology groups $\mathrm{H}_n(X, A) = \mathrm{H}_{n+1}(X, A) = 0$, i.e., there is an exact sequence

$$\ldots \to 0 \to \mathrm{H}_n(A) \to \mathrm{H}_n(X) \to 0 \to \ldots$$

which means that the map in the middle is an isomorphism.

Example 4.32. Let $X = B(R, x) \subset \mathbf{R}^n$ be an open (non-empty) ball, with $n > 0$. Then the so-called *local homology groups* are isomorphic to:

$$\mathrm{H}_k(X, X \backslash \{x\}) \cong \begin{cases} \mathbf{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

Here, the group $\mathbf{Z}$ stems from the group $\mathrm{H}^{n-1}(S^{n-1})$, where S^{n-1} is a little $(n-1)$-sphere around x.

Indeed, X, being convex, has $\mathrm{H}_k(X) = 0$ for $k > 0$. Also, $X \backslash \{x\}$ is homeomorphic to $S^{n-1} \times \mathbf{R}$ and therefore homotopy equivalent to S^{n-1}. (We say two spaces X, $Y \in \mathrm{Top}$ are *homotopy equivalent* if there is a (continuous) homotopy equivalence $f : X \to Y$.)

Therefore, the above long exact sequences reads

$$\ldots \to \mathrm{H}_k(S^{n-1}) \xrightarrow{i_k} \underbrace{\mathrm{H}_k(X)}_{=0 (k>0)} \to \mathrm{H}_k(X, X\backslash\{x\}) \xrightarrow{d} \mathrm{H}_{k-1}(S^{n-1}) \xrightarrow{i_{k-1}} \underbrace{\mathrm{H}_{k-1}(X)}_{=0 (k>1)}$$

so the connecting homomorphism d is an isomorphism except for $k > 1$. For $k = 0$, the map i_0 identifies with the identity map $\mathbf{Z} \to \mathbf{Z}$, so that d is injective, and hence the local homology group vanishes. For $k = 1$, the group $\mathrm{H}_k(X) = 0$, so that our group is $\ker i_0 = 0$.

Here is another bread-and-butter result from homological algebra.

Lemma 4.33. (*Five lemma*) Let

$$\begin{tikzcd}
C_5 \arrow[r] \arrow[d, "f_5"] & C_4 \arrow[r] \arrow[d, "f_4"] & C_3 \arrow[r] \arrow[d, "f_3"] & C_2 \arrow[r] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\
D_5 \arrow[r] & D_4 \arrow[r] & D_3 \arrow[r] & D_2 \arrow[r] & D_1
\end{tikzcd}$$

be a map between two *exact* chain complexes (of abelian groups or, more generally, objects in any abelian category). Suppose that f_2 and f_4 are isomorphisms, f_1 is injective, and f_5 is surjective. Then f_3 is an isomorphism.

Proof. The proof is a typical case of diagram-chasing, see [Stacks, Tag 05QB]. (To show f_3 is surjective, one only needs f_2 and f_4 surjective and f_1 injective.) $\square$

Theorem 4.34. (*Excision*) Let $Z, A \subset X$ be two subspaces such that

$$\overline{Z} \subset A^\circ$$

(closure and interior, respectively). Then there is a natural isomorphism

$$\mathrm{H}_n(X\backslash Z, A\backslash Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

Proof. Putting $B := X\backslash Z$ we have $X\backslash\overline{Z} = B^\circ$. Thus, by assumption, we get a covering

$$A^\circ \cup B^\circ = X.$$

Let us write $C(A+B) := C^{\mathcal{U}}(X)$ for this covering. We are getting short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C(A\cap B) \arrow[r] \arrow[d] & C(B) \arrow[r] \arrow[d] & C(B)/C(A\cap B) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & C(A) \arrow[r] \arrow[d, equal] & C(A+B) \arrow[r] \arrow[d, "\sim"] & C(A+B)/C(A) \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & C(A) \arrow[r] & C(X) \arrow[r] & C(X)/C(A) \arrow[r] & 0.
\end{tikzcd}$$

The maps labelled "$\sim$" are quasi-isomorphisms: for the middle map this is Theorem 4.19, and for the right hand map this then follows from the five lemma above, applied to the long exact homology sequences provided by the snake lemma.

In each chain degree, the top left square is a pushout square of abelian groups (cf. the proof of Lemma 4.17), so that the right hand vertical map is an isomorphism (in each chain degree, and therefore, since it is a chain map, also a chain isomorphism). □

Example 4.35. Recall that a *topological manifold* of dimension n is a topological space X such that every point $x \in X$ has an open neighborhood that is homeomorphic to an open ball in $\mathbf{R}^n$. For such a manifold, we can now strengthen the above computation of local homology: for any $x \in X$ and any such open neighborhood $U \ni x$ we have isomorphisms

$$\mathrm{H}_k(X, X\backslash\{x\}) = \mathrm{H}_k(U, U\backslash\{x\}) = \begin{cases} \mathbf{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

Indeed the first isomorphism follows by taking $Z := X\backslash U$ and $A = X\backslash\{x\}$ in Theorem 4.34.

Outlook 4.36. While the excision isomorphism above is canonical (i.e., functorial with respect to inclusions $U \subset X$), this is not the case for the right hand isomorphism. It is therefore *not* in general possible to choose these isomorphisms in a way that is compatible for all U. A manifold is called *orientable* if this is in fact possible. We will study this matter more in depth using cohomological methods, and for the moment just state that $\mathbf{RP}^n$ is *not orientable*, while S^n and $\mathbf{CP}^n$ and more generally, all complex manifolds, are orientable.

4.6 The mapping degree

We will leverage our understanding of how continuous maps act on $\mathrm{H}_1(S^1)$ in order to prove the hedgehog theorem and the fundamental theorem of algebra. Before that, we systematize our considerations above a bit.

Definition and Lemma 4.37. Let $k \geqslant 1$ and $f : S^k \to S^k$ a continuous map. Then $\deg f$ is the unique integer such that the map

$$f_* : \mathrm{H}_k(S^k) \to \mathrm{H}_k(S^k)$$

is multiplication by $\deg(f)$.

This map has the following properties:

(1) The assignment $f \mapsto \deg f$ can be organized into a map

$$\mathrm{End}_{\mathrm{HoTop}}(S^k) \stackrel{\deg}{\to} \mathbf{Z},$$

i.e., homotopic maps have the same degree.

(2) deg is a monoid homomorphism, i.e., $\deg(\mathrm{id}_{S^k}) = 1$ and

$$\deg(g \circ f) = \deg(g) \cdot \deg \deg(f).$$

(3) The degree of a constant map is 0.

(4) The degree of a reflection r (along a hyperplane through the origin) is $\deg r = -1$.

(5) The degree of $i : z \mapsto -z$ is $\deg i = (-1)^{k+1}$.

(6) For $k = 1$, the degree of $z \mapsto z^d$ (for $z \in S^1 \subset \mathbf{C}$) is d, for any $d \in \mathbf{Z}$.

Proof. Any group homomorphism $\mathbf{Z} \to \mathbf{Z}$ is multiplication by a unique integer d. The existence and unicity of $\deg f$ then follows from the isomorphism

$$\mathbf{Z} \to \mathrm{H}_k(S^k).$$

The remaining statements hold by functoriality of H_k and the homotopy axiom. The third statement holds since f factors as $S^k \to \{*\} \to S^k$, and thus $\mathrm{H}_k(f) : \mathrm{H}_k(S^k) \to \mathrm{H}_k(S^0) = 0 \to \mathrm{H}_k(S^k)$ must vanish.

Sketch of (4): Any reflection is homotopic to the map $\iota : (x_0, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$. Tracing down the isomorphisms $\mathrm{H}_k(S^k) \cong \mathrm{H}_{k-1}(S^{k-1})$, one reduces the computation $\deg \iota = -1$ to the case $k = 1$, which is part of Example 4.63.

(5) then follows since i is the composition of $k+1$ reflections. (6) is proven in Example 4.63. (Alternatively, instead of using Example 4.63, one can also prove that the map $\pi_1(S^1, *) \to \mathrm{H}_1(S^1)$ mentioned in Remark 4.6 is a group homomorphism. In $\pi_1(S^1)$, the loop winding around n times is the n-fold sum of loops winding around once.) $\square$

Outlook 4.38. For categorical thinkers, the map deg is just the evaluation of the functor $\mathrm{H}_k : \mathrm{HoTop} \to \mathrm{Ab}$:

$$\mathrm{End}_{\mathrm{HoTop}}(S^k) \to \mathrm{End}_{\mathrm{Ab}}(\mathrm{H}_k(S^k)) = \mathrm{End}_{\mathrm{Ab}}(\mathbf{Z}) = \mathbf{Z}.$$

The right-most isomorphism maps an $n \in \mathbf{Z}$ to the map $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$ (multiplication by n).

The rôle of S^k is not that special in the definition of the mapping degree. Poincaré duality asserts (among other things), that for a compact connected orientable manifold M of dimension k, $\mathrm{H}_k(M) \cong (\mathrm{H}_0(M))^\vee = \mathbf{Z}^\vee = \mathbf{Z}$, and then the definition above carries over verbatim.

Our knowledge about the mapping degree has various consequences such as the hedgehog theorem and the fundamental theorem of algebra.

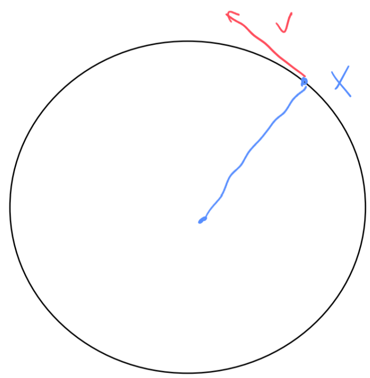
Definition 4.39. The *tangent bundle* of S^n is

$$TS^n := \{(x, v) \in S^n \times \mathbf{R}^{n+1} | \langle x, v \rangle = 0\}.$$

Here the equation $\langle x, v \rangle = 0$ signifies that v is a tangent vector at S^n at the point x. The tangent bundle comes with a natural map

$$\pi : TS^n \to S^n, (x, v) \mapsto x.$$

A *vector field* is a (continuous) section of this map, i.e., a map of the form $x \mapsto (x, v(x))$.

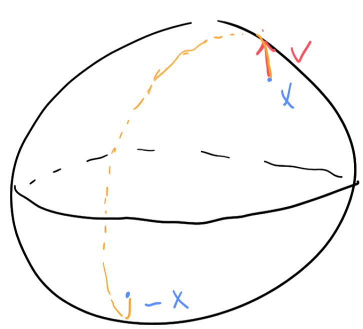


Corollary 4.40. (*Hedgehog theorem* or *hairy ball theorem*) For $n \geqslant 1$ there is a non-vanishing (continuous!) vector field on S^n if and only if n is odd.

Proof. If n is odd, then

$$v(x) := (x_2, -x_1, x_4, -x_3, \dots, x_{n+1}, -x_n)$$

provides a non-vanishing vector field. Suppose, conversely, that v is a non-vanishing vector field on S^n. Let $h : S^n \times [0,1] \to S^n$ be the *geodesic* from x to $-x$ in the direction of $v(x)$.



In a formula,

$$h(x,t) := \cos(\pi t)x + \sin(\pi t)\frac{v(x)}{||v(x)||}.$$

Then h is a homotopy between id_{S^n} and $-\mathrm{id}_{S^n}$. Thus, the mapping degrees

$$1 = \deg(\mathrm{id}_{S^n}) = \deg(-\mathrm{id}_{S^n}) = (-1)^{n+1}$$

so that n is odd. $\square$

Corollary 4.41. (*Fundamental theorem of algebra*) Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a non-constant complex polynomial (i.e., $a_k \in \mathbf{C}$, and $a_n \neq 0$ for $n > 0$). Then there is some $z_0 \in \mathbf{C}$ such that

$$f(z_0) = 0.$$

Proof. We may assume $a_n = 1$, for notational simplicity. We assume f has no zeros, and hence is a continuous map

$$f : \mathbf{C} \to \mathbf{C}^\times := \mathbf{C} \backslash \{0\}.$$

The overall idea of the proof is to construct maps that have mapping degree n, but, if f has no zeros, deform this map into one that has degree 0.

For $R > 0$, we consider the polynomial

$$f_R(z) := R^n f(z/R) = z^n + \frac{a_{n-1}}{R} z^{n_1} + \cdots + \frac{a_0}{R^n}.$$

Since f has no zeros, nor does f_R have any zeros, so we can set

$$g_R(z) := \frac{f_R(z)}{|f_R(z)|}.$$

(1) For $R \gg 0$, this is getting close to the polynomial z^n in the sense that for $R \gg 0$

$$|f_R(z) - z^n| < 1 \tag{4.42}$$

for all $z \in S^1$. This implies that $f_R : S^1 \to \mathbf{C}^\times$ is homotopic to the map $z \mapsto z^n$. Indeed,

$$h(z, t) := t f_R(z) + (1 - t) z^n$$

is a continuous map $S^1 \times [0, 1] \to \mathbf{C}$. It is in fact taking values in $\mathbf{C}^\times$: if $h(z, t) = 0$, then $t \neq 0$ and $f_R(z) = \frac{t-1}{t} z^n$, and hence $|f_R(z) - z^n| = \frac{1}{t} \geqslant 1$, contradicting (4.42).

(2) For $R \gg 0$, the homotopy $f_R \sim z^n$ gives rise to a homotopy of g_R to the map $z \mapsto z^n$. The degree of the latter map is n.

(3) On the other hand, the map $f = f_1$ is homotopic to f_R (both are regarded as maps $S^1 \to \mathbf{C}^\times$) and therefore $g := g_1$ is homotopic to g_R (both are maps $S^1 \to S^1$) so that

$$\deg g = \deg g_R.$$

The map g so-defined admits an extension to the closed ball $\overline{B}^2$, namely $\frac{f(z)}{|f(z)|}$. Here we use critically that f has no zeros.

Thus,

$$g_* : \mathrm{H}^1(S^1) \to \mathrm{H}^1(\overline{B}^2) = 0 \to \mathrm{H}^1(S^1),$$

so that $\deg g = 0$.

(4) We obtain $n = \deg g_R = \deg g = 0$, contradicting the assumption that f is non-constant. $\square$

Points (5) and (6) in Definition and Lemma 4.37 suggest that the degree of a map $f : S^k \to S^k$ is related to the number of preimages of a given point. This is indeed so, provided we count the preimages in the right way, i.e., with appropriate multiplicities. These multiplicities are the local degrees of f:

Definition 4.43. Let $f : S^k \to S^k$ be a map, $y \in S^k$ in the codomain, and suppose that $f^{-1} = \{x_1, \dots, x_m\}$. Then we can choose neighborhoods $V \ni y$ and $U_i \ni x_i$ such that $f(U_i \backslash \{x_i\}) \subset V \backslash \{y\}$. For clarity, write $f_i := f|_{U_i}$. The *local degree* of f at x_i, denoted by $\deg_{x_i} f$ is the integer such that the bottom horizontal map, which is defined to be the one making the diagram commutative, is multiplication by $\deg_{x_i} f$:

$$\begin{tikzcd}
\mathrm{H}_k(U_i, U_i\backslash\{x_i\}) \arrow[r, "(f_i)_*"] \arrow[d, "\cong"] & \mathrm{H}_k(V, V\backslash\{y\}) \arrow[d, "\cong"] \\
\mathrm{H}_k(S^k, S^k\backslash\{x_i\}) & \mathrm{H}_k(S^k, S^k\backslash\{y\}) \\
\mathrm{H}_k(S^k) \arrow[u, "\cong"] \arrow[r] & \mathrm{H}_k(S^k). \arrow[u, "\cong"]
\end{tikzcd}$$

Here the two upper vertical maps are excision isomorphisms, while the lower ones come from the long exact sequence (4.31).

In the above diagram, the bottom horizontal map is *not* in general the map induced by f, as the following lemma shows. (The problem is that f does not induce a map $S_k\backslash\{x_i\} \to S_k\backslash\{y\}$, if $f^{-1}(y) \supsetneq \{x_i\}$, so that in the above diagram one can not insert a natural map f_* in the middle row.)

Lemma 4.44. In the above situation, one has

$$\deg f = \sum_{i=1}^{m} \deg_{x_i} f.$$

Proof. Pick disjoint neighborhoods $U_i \ni x_i$ that are mapped by f into a neighborhood $V \ni y$. □

4.7 Cellular homology

We have defined the homology of simplicial sets and have computed, without much ado, the homology of the simplicial k-sphere

$$\mathrm{H}_k(S^n_{\mathrm{simp}}) = \begin{cases} \mathbf{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

Using the Mayer–Vietoris sequence, we have been able to compute $\mathrm{H}_k(S^n_{\mathrm{Top}})$, and it turns out that

$$\mathrm{H}_k(S^n_{\mathrm{Top}}) = \mathrm{H}_k(S^n).$$

The topological sphere arises as the pushout

$$\begin{tikzcd} \partial\Delta^n_{\mathrm{Top}} \arrow[r] \arrow[d] & \Delta^n_{\mathrm{Top}} \arrow[d] \\ \Delta^0 \arrow[r] & S^n_{\mathrm{Top}}, \end{tikzcd}$$

which is very much the same as the correpsonding diagram for the simplicial n-sphere. (In fact, $|S^n_{\mathrm{simp}}|$ is homeomorphic to S^n_{Top}.) The process of glueing in "cells" of higher dimension (in this case, glueing in an n-simplex Δ^n) along its boundary into an already existing space (in this case Δ^0) is quite wide-spread. In this section we study the homology of such spaces systematically.

Definition 4.45. A *cell complex* or *CW complex* is a topological space $X = \bigcup_{k \geqslant 0} X_k$ such that

- X_0 is a finite discrete topological space,
- X_k is obtained from X_{k-1} by attaching finitely many k-cells, i.e., there is a pushout diagram (for a finite, possibly empty,

set J_k)

$$\begin{tikzcd} \bigsqcup_{j \in J_k} \partial \overline{B}^k \arrow[r] \arrow[d] & X_{k-1} \arrow[d] \\ \bigsqcup_{j \in J_k} \overline{B}^k \arrow[r] & X_k. \end{tikzcd}$$

Here $\overline{B}^k$ denotes a closed (non-empty) ball in $\mathbf{R}^k$ and the left vertical maps are the inclusions of the boundary.

- The topology on X is the *weak topology*: a $U \subset X$ is open iff all the $U \cap D^k$ are open. (Equivalently, X is the colimit of the diagram $X_0 \to X_1 \to \dots$ in the category Top.)

Example 4.46.
- The pushout

$$\begin{tikzcd} \partial \overline{B}^k \arrow[r] \arrow[d] & X_0 := \{*\} \arrow[d] \\ \overline{B}^k \arrow[r] & S^k \end{tikzcd}$$

shows that S^k is a CW complex. (Note the similarity to the simplicial sphere, cf. Exercise 2.7!)

- $S^n \times S^m$ is a CW complex with one cell in dimensions 0, n, m and $n + m$. For example, the torus $T^2 = S^1 \times S^1$ has cells in dimension 0 (one), 1 (two), 2 (one).

- Recall that the *real projective space* is defined as

$$\begin{aligned} \mathbf{RP}^n &:= \left(\mathbf{R}^{n+1}\backslash\{0\}\right)/x \sim \lambda x \text{ for } \lambda \in \mathbf{R}\backslash\{0\} \\ &= S^n/(x \sim -x) \end{aligned}$$

and is equipped with the quotient topology. The inclusions $S^{n-1} \subset S^n$ at the equator $(x_0, \dots, x_{n-1}) \to (x_0, \dots, x_{n-1}, 0)$ are compatible with these identifications and show that $\mathbf{RP}^n$ is a cell complex with exactly one cell in dimensions $0, \dots, n$.

- *Complex projective space* is defined as

$$\begin{aligned} \mathbf{CP}^n &:= \left(\mathbf{C}^{n+1}\backslash\{0\}\right)/x \sim \lambda x \text{ for } \lambda \in \mathbf{C}\backslash\{0\} \\ &= S^{2n+1}/x \sim \lambda x \text{ for } |\lambda| = 1\}. \end{aligned}$$

There is a homeomorphism

$$\begin{aligned} S^{2n+1}/x \sim \lambda x &= \{(w, \sqrt{1-|w|^2}) \in \mathbf{C}^{n+1}, |w| \leqslant 1\}/(w,0) \sim \lambda(w,0) \text{ for } |w| = 1 \\ &= \overline{B}^{2n}/w \sim \lambda w \text{ for } w \in \partial \overline{B}^{2n}. \end{aligned}$$ $\blacksquare$

Since $\partial \overline{B}^{2n} = S^{2n-1}$, this shows that we have a pushout diagram

$$\begin{tikzcd} \partial \overline{B}^{2n} \arrow[r] \arrow[d] & \mathbf{CP}^{n-1} \arrow[d] \\ \overline{B}^{2n} \arrow[r] & \mathbf{CP}^n. \end{tikzcd}$$

Therefore, $\mathbf{CP}^n$ is a cell complex with one cell in dimensions $0, 2, \ldots, 2n$. In (complex) dimension 1, $\mathbf{CP}^1 \cong S^2$ is also called the *Riemann sphere*.

- Infinite real and complex projective spaces are defined as

$$\mathbf{RP}^\infty := \bigcup_{n \geqslant 0} \mathbf{RP}^n, \mathbf{CP}^\infty := \bigcup_{n \geqslant 0} \mathbf{CP}^n.$$

They are cell complexes with one cell in each dimension, resp. in each even dimension.

Definition 4.47. For a topological space Y, the *reduced chain complex* $\widetilde{C}(Y)$ is defined by

$$\widetilde{C}(Y)_n := \begin{cases} C_n(Y) & n > 0 \\ \ker C_0(Y) \to C_0(\{*\}) = \mathbf{Z} & n = 0 \end{cases}.$$

(The map in degree 0 is applying C_0 to the map $Y \to \{*\}$, i.e., $\sum n_y y \mapsto \sum n_y$.)

This is (!)indeed a chain complex (the idea of appending a $\mathbf{Z}$ at the end already appeared in the proof of Theorem 4.19). We define the *reduced homology*

$$\widetilde{\mathrm{H}}_n(Y) := \mathrm{H}_n(\widetilde{C}(Y)) = \begin{cases} \mathrm{H}_n(Y) & n > 0 \\ \ker \mathrm{H}_0(Y) \to \mathrm{H}_0(\{*\}) = \mathbf{Z} & n = 0 \end{cases}.$$

For example, if Y is connected, the group $\widetilde{\mathrm{H}}_0(Y) = 0$.

Let $y \in Y$ be a point, and write $i : \{y\} \to Y$ for the inclusion. There is an exact sequence

$$0 \to \mathrm{H}_0(\{y\}) \xrightarrow{i_*} \mathrm{H}_0(Y) \to \mathrm{H}_0(Y, y) \to 0.$$

Except for the zero at the left, this is a consequence of the definition of relative homology. The left hand map $i_*(:= \mathrm{H}_0(i))$ is injective since the map $p : Y \to \{y\}$ (sending everything to that point) satisfies $p \circ i = \mathrm{id}$, so that $p_* \circ i_* = \mathrm{id}$. Thus, the sequence splits, i.e., there is an isomorphism

$$\mathrm{H}_0(Y) \cong \mathbf{Z} \oplus \mathrm{H}_0(Y, y).$$

For the same reason, the exact sequence

$$0 \to \widetilde{\mathrm{H}}_0(Y) \to \mathrm{H}_0(Y) \xrightarrow{p_*} \mathrm{H}_0(\{y\}) \to 0$$

splits, and we obtain an isomorphism

$$\widetilde{\mathrm{H}}_0(Y) \cong \mathrm{H}_0(Y, y). \tag{4.48}$$

Definition 4.49. A *good subspace* $A \subset X$ of a topological space X is such that A is closed and that there is a neighborhood V of A in X such that the inclusion $A \subset V$ is a deformation retract (Example 4.11).

Proposition 4.50. If $A \subset X$ is a good subspace, the quotient map

$$q : (X, A) \to (X/A, A/A)$$

induces is an isomorphism

$$q_* : \mathrm{H}_n(X, A) \cong \widetilde{\mathrm{H}}_n(X/A).$$

Proof. Let $V \supset A$ be a neighborhood as in Definition 4.49: then the inclusion $A \subset V$ includes isomorphisms $\mathrm{H}_*(A) \xrightarrow{\cong} \mathrm{H}_*(V)$ and thus the five lemma yields long exact sequence for relative homology gives the isomorphisms marked $*$. Again using the deformation retract $A \subset V$, V/A is homotopy equivalent to $A/A \cong \{*\}$, so that we get the isomorphisms marked $**$:

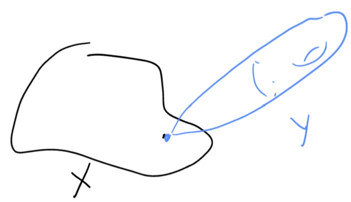
$$\begin{tikzcd}
\mathrm{H}_*(X, A) \arrow[r, "*", "\cong"'] \arrow[d] & \mathrm{H}_*(X, V) \arrow[d] & \mathrm{H}_*(X - A, V - A) \arrow[l, "\cong"', "Exc."] \arrow[d, "\cong"] \\
\mathrm{H}_*(X/A, \{*\}) \arrow[r, "**", "\cong"'] & \mathrm{H}_*(X/A, V/A) & \mathrm{H}_*(X/A - A/A, V/A - A/A). \arrow[l, "\cong", "Exc."']
\end{tikzcd}$$

The two horizontal isomorphism at the right are excision isomorphisms. The vertical maps arise via q_*. Since that map q is a homeomorphism on the complement $X\backslash A$, it gives the right hand vertical isomorphism. Thus the left hand vertical map is also an isomorphism. We conclude using $\mathrm{H}(X/A, \{*\}) = \widetilde{\mathrm{H}}(X/A)$, cf. (4.48).□

In order to compute the homology of cell complexes, we need a little preparation:

Definition 4.51. If $X, Y \in \mathrm{Top}$ are topological spaces with base points x, y, the *wedge sum* is defined by

$$X \vee Y := X \sqcup Y/x \sim y.$$



The same definition applies for possibly infinitely many pointed spaces (X_i, x_i):

$$\bigvee_{i \in I} X_i := \bigsqcup X_i/x_i \sim x_j. \tag{4.52}$$

Lemma 4.53. In the situation of (4.52), suppose that the $\{x_i\} \subset X_i$ are good subspaces. Then there are isomorphisms

$$\bigoplus_{i \in I} \widetilde{\mathrm{H}}_*(X_i) \xrightarrow{\cong} \widetilde{\mathrm{H}}_*(\bigvee_i X_i).$$

Proof. We have isomorphisms

$$\bigoplus_i \widetilde{\mathrm{H}}_*(X_i) = \bigoplus_i \mathrm{H}_*(X_i, \{x_i\}) = \mathrm{H}_*(\bigsqcup X_i, \bigsqcup \{x_i\}) = \widetilde{\mathrm{H}}_*(\bigvee_i X_i),$$

by Proposition 4.50, the additivity axiom (extended to relative homology), and again Proposition 4.50, where we use that the inclusion $\bigsqcup\{x_i\} \subset \bigsqcup X_i$ is a good subspace, as well. □

Proposition 4.54. Let $X = \bigcup_k X_k$ be a cell complex. Let n_k be the number of disks of dimension k being attached (i.e., $n_0 = |X_0|$ and $n_k = |J_k|$ in Definition 4.45). There holds:

- $$\mathrm{H}_n(X_k, X_{k-1}) = \begin{cases} \mathbf{Z}^{n_k} & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.55}$$

- The homology groups of X in low degree are controlled by the low-dimensional pieces of a cell complex. More formally, the inclusion $X_k \to X$ induces an isomorphism
$$\mathrm{H}_n(X_k) \stackrel{\cong}{\to} \mathrm{H}_n(X) \text{ for } n < k. \tag{4.56}$$

- We also have $\mathrm{H}_n(X_k) = 0$ for $n > k$.

Proof. The inclusion $X_{k-1} \subset X_k$ is a good subspace, so that
$$\mathrm{H}_*(X_k, X_{k-1}) = \widetilde{\mathrm{H}}_*(X_k/X_{k-1}) \cong \widetilde{\mathrm{H}}_*(\bigvee_{n_k} S^k)$$
which by Definition 4.51 (and the computation of $\mathrm{H}_*(S^k)$!) takes the value stated above.

The idea of the second point is to use that the (homological) difference between X_k and X_{k-1} just lives in degree k. More formally, for the very last assertion, we use the long exact sequence
$$\ldots \to \mathrm{H}_{n+1}(X_k, X_{k-1}) \to \mathrm{H}_n(X_{k-1}) \to \mathrm{H}_n(X_k) \to \mathrm{H}_n(X_k, X_{k-1}) \to \ldots$$ ∎

- The outer groups are zero for $n > k$, giving an isomorphism in the middle. Thus, $\mathrm{H}_n(X_k) = \mathrm{H}_n(X_0) = 0$ for $n > k \geqslant 0$.

- The outer groups are also zero for $n < k - 1$. If $X = X_N$ for $N \gg 0$ this immediately shows the second assertion.

- We now prove that, in general, the isomorphisms $\mathrm{H}_n(X_{k-1}) \stackrel{\cong}{\to} \mathrm{H}_n(X_k)$ yield the isomorphism (4.56). We have that
$$C(X) = \bigcup_{k \geqslant 0} C(X_k).$$
(Note that $C(X_k) \subset C(X)$ is a subcomplex since $X_k \subset X$ is a subspace.) I.e., every element $f = \sum n_\sigma \sigma \in C(X)_r$ comes from some X_k, for large enough k (depending on f). To see this, it suffices to consider some $\sigma : \Delta^r \to X = \bigcup X_k$. The inclusions $X_k \subset X$ are *closed* subspaces and Δ^r is compact. Thus, we can find a finite subcovering of the open covering $\Delta^r = \sigma^{-1}(\bigcup_k X \backslash X_k)$, i.e., $\sigma(\Delta^r) \subset X_k$ for large k.

With this topological preparation, let f be a n-cycle in X. It is an n-cycle in X_k for $k \gg 0$. Thus, $[f]$ lies in the image of $\mathrm{H}_n(X_k) \to \mathrm{H}_n(X)$, which by the previous step (applied to X_k instead of X) is the same as the image of $\mathrm{H}_n(X_{n+1})$. For the injectivity, suppose an n-cycle f in X_k is a boundary on X. This boundary comes from some X_K for $K \gg k$, so that f is zero in the homology of X_K. Again applying the finite-dimensional case (to X_K), $f = 0$ as well. $\square$

Definition and Lemma 4.57. Let $X = \bigcup X_k$ be a cell complex. The *cellular chain complex* $C^{\mathrm{cell}}(X)$ is defined by

$$C^{\mathrm{cell}}(X)_k := \mathrm{H}_k(X_k, X_{k-1})$$

and differential

$$\partial_k^{\mathrm{cell}} : \mathrm{H}_k(X_k, X_{k-1}) \to \mathrm{H}_{k-1}(X_{k-1}) \to \mathrm{H}_{k-1}(X_{k-1}, X_{k-2}),$$

i.e., the composite of the indicated maps in the long exact sequences (4.31). This is indeed a complex, i.e., $\partial_k^{\mathrm{cell}} \circ \partial_{k+1}^{\mathrm{cell}} = 0$, so we can define the *cellular homology* of X as

$$\mathrm{H}_*^{\mathrm{cell}}(X) := \mathrm{H}_*(C^{\mathrm{cell}}(X)).$$

Proof. This holds since the two maps going down-right compose to zero:

$$\begin{tikzcd}
\mathrm{H}_k(X_k) \arrow[dr] & & & \\
\mathrm{H}_{k+1}(X_{k+1}, X_k) \arrow[u] \arrow[r, "\partial_{k+1}^{\mathrm{cell}}"] & \mathrm{H}_k(X_k, X_{k-1}) \arrow[r, "\partial_k^{\mathrm{cell}}"] \arrow[dr] & \mathrm{H}_{k-1}(X_{k-1}, X_{k-2}) \\
& & \mathrm{H}_{k-1}(X_{k-1}) \arrow[u]
\end{tikzcd}$$

Example 4.58. For $X = S^n$, with the above cell structure ($X_0 = \{*\} = X_1 = \cdots = X_{n-1} \subset X_n = S^n$), we have

$$C^{\mathrm{cell}}(S^n) = [\mathbf{Z} \to 0 \to \ldots \to 0 \to \mathbf{Z}].$$

The differentials are all zero. For $n > 1$ this is clear and for $n = 1$ we note that the map $\mathrm{H}_1(S^1, \{*\}) \to \mathrm{H}_0(\{*\})$ (and therefore also $\partial_1^{\mathrm{cell}}$) is zero since it lies in the exact sequence

$$\ldots \to \mathrm{H}_1(S^1, \{*\}) \to \mathrm{H}_0(\{*\}) = \mathbf{Z} \xrightarrow{\mathrm{id}} \mathrm{H}_0(S^1) = \mathbf{Z}.$$

Thus, the above complex is the *same* complex as $N(\mathbf{Z}[S^n_{\mathrm{simp}}])$.

In order to compute more general examples, we need the following formula for the cellular differential. Recall from Definition 4.45 that J_k denotes the set of k-cells glued in by passing from X_{k-1} to X_k. For $i \in J_k$, we denote the corresponding generator of

$$\mathrm{H}_k(X_k, X_{k-1}) = \bigoplus_{i \in J_k} \mathbf{Z}$$

by e_i.

Lemma 4.59. The cellular differential is given by

$$\partial^{\mathrm{cell}}(e_i) = \sum_{j \in J_{k-1}} d_{ij} e_r,$$

where d_{ij} is the mapping degree of the following map

$$S_i^{k-1} \xrightarrow{a_i} X_{k-1} \to X_{k-1}/X_{k-2} = \bigvee_{j \in J_{k-1}} S^{k-1} \xrightarrow{q_j} S^{k-1}.$$

The map a_i is the map $S_i^{k-1} = \partial \overline{B}^k \to X_{k-1}$ that is part of the definition of a cell complex. The right map collapses all copies of S^{k-1} different from j to a point. (The sum is finite since the copy of $\overline{B}^k$ corresponding to i has compact image and therefore only intersects finitely many $(k-1)$-cells.)

Proof. This basically follows from the definitions. The cell differential is the diagonal map in the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_k(\overline{B}^k_i, \partial \overline{B}^k_i) \arrow[r, "\partial", "\cong"'] \arrow[d, "a_i"] & \widetilde{\mathrm{H}}_{k-1}(\partial \overline{B}^k_i) \arrow[r, "\Delta"] \arrow[d, "a_i"] & \widetilde{\mathrm{H}}_{k-1}(S^{k-1}_j) \\
\mathrm{H}_k(X_k, X_{k-1}) \arrow[r, "\partial"] \arrow[dr] & \widetilde{\mathrm{H}}_{k-1}(X_{k-1}) \arrow[r, "q"] \arrow[d, leftrightarrow, "q_j", "\cong"'] & \widetilde{\mathrm{H}}_{k-1}(X_{k-1}/X_{k-2}) \arrow[u, "q_j"'] \arrow[d, "\cong"] \\
 & \mathrm{H}_{k-1}(X_{k-1}, X_{k-2}) \arrow[r] & \widetilde{\mathrm{H}}_{k-1}(X_{k-1}/X_{k-2}, X_{k-2}/X_{k-2}).
\end{tikzcd}$$

The two maps labelled "a_i" arise from glueing in the k-cell corresponding to i, i.e., the map $a_i : \partial \overline{B}^k \to X_{k-1}$. Under the above computations, the left vertical map is the inclusion $\mathbf{Z} \to \bigoplus_{i \in J_k} \mathbf{Z}$

into the i-th copy of $\mathbf{Z}$. The two maps labelled "∂" are boundary maps of the long exact sequence of relative homology. The map q arises from the quotient map, q_j is as in the claim, so that Δ is the pushforward along $q_j q a_i$. □

Example 4.60. We compute the cellular homology of $X = \mathbf{RP}^n$, using the above cell structure (Example 4.46). We have one cell in each dimension $k \leqslant n$ and the attaching map is the standard quotient map $S^{k-1} \to X_{k-1} := \mathbf{RP}^{k-1}$. The cell differential is the composite

$$\begin{aligned} \mathrm{H}_k(\mathbf{RP}^k, \mathbf{RP}^{k-1}) = \mathrm{H}_k(\mathbf{RP}^k/\mathbf{RP}^{k-1}) &= \mathrm{H}_k(S^k) \\ &= \mathrm{H}_{k-1}(S^{k-1}) \\ &\xrightarrow{p_*} \mathrm{H}_{k-1}(\mathbf{RP}^{k-1}) \\ &\xrightarrow{q_*} \mathrm{H}_{k-1}(\mathbf{RP}^{k-1}/\mathbf{RP}^{k-2}) \\ &= \mathrm{H}_{k-1}(S^{k-1}), \end{aligned}$$

where

$$S^{k-1} \xrightarrow{p} \mathbf{RP}^{k-1} \xrightarrow{q} \mathbf{RP}^{k-1}/\mathbf{RP}^{k-2} = S^{k-1}$$

are the degree 2 covering and the projection onto the quotient, respectively. The composite qp is a homeomorphism on each of the two components of $S^{k-1} \backslash S^{k-2}$. These two homeomorphisms are obtained from each other as antipodes. The degree of the antipode map is $(-1)^k$. By Lemma 4.44, we can compute the degree as the sum of local degrees, i.e.,

$$\deg qp = 1 + (-1)^k.$$

Thus the cell complex is concentrated in degrees $n, \ldots, 0$ and reads

$$0 \to \mathbf{Z} \xrightarrow{2 \text{ or } 0} \mathbf{Z} \to \ldots \to \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \to 0.$$

We obtain

$$\mathrm{H}_k^{\mathrm{cell}}(\mathbf{RP}^n) = \begin{cases} \mathbf{Z} & k = 0 \\ \mathbf{Z}/2 & 0 < k < n,\ k \text{ odd} \\ \mathbf{Z} & k = n \text{ for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Note that the groups $C^{\mathrm{cell}}(X)_k$ depend on the way X is presented as a cell complex, i.e., the choice of the X_k. However, its homology does not, as we now see.

Theorem 4.61. Cellular homology agrees with singular homology. More formally, for a cell complex $X = \bigcup_k X_k$, there is an isomorphism

$$\mathrm{H}_*(X) \xrightarrow{\cong} \mathrm{H}_*^{\mathrm{cell}}(X).$$

Proof. We consider the commutative diagram of (relative) homology groups

$$\begin{tikzcd}
 & & & 0 = \mathrm{H}_k(X_{k+1}, X_k) & \\
0 = \mathrm{H}_k(X_{k-1}) \arrow[dr] & & \mathrm{H}_k(X_{k+1}) = \mathrm{H}_k(X) \arrow[ur] & & \\
 & \mathrm{H}_k(X_k) \arrow[ur] \arrow[dr, "\varphi"] & & & \\
\mathrm{H}_{k+1}(X_{k+1}, X_k) \arrow[ur, "d"] \arrow[rr, "\partial_{k+1}^{\mathrm{cell}}"] & & \mathrm{H}_k(X_k, X_{k-1}) \arrow[rr, "\partial_k^{\mathrm{cell}}"] \arrow[dr, "\psi"] & & \mathrm{H}_{k-1}(X_{k-1}, X_{k-2}) \\
 & & & \mathrm{H}_{k-1}(X_{k-1}) \arrow[ur, "i"] & \\
 & & 0 = \mathrm{H}_{k-1}(X_{k-2}) \arrow[ur] & &
\end{tikzcd}$$

The diagonal exact sequences are the ones from (4.31), the diagram commutes by definition of the cellular complex, and the vanishings hold by Proposition 4.54. Thus, we have isomorphisms

$$\begin{aligned}
\mathrm{H}_k(X) &= \mathrm{H}_k(X_{k+1}) \\
&= \operatorname{coker} d \\
&= \operatorname{im} \varphi / \operatorname{im} \partial_{k+1}^{\mathrm{cell}} \text{ by the injectivity of } \varphi \\
&= \ker \psi / \operatorname{im} \partial_{k+1}^{\mathrm{cell}} \text{ by the exactness of the down-right diagonal} \\
&= \ker \partial_k^{\mathrm{cell}} / \operatorname{im} \partial_{k+1}^{\mathrm{cell}} \text{ by the injectivity of } i \\
&=: \mathrm{H}_k^{\mathrm{cell}}(X).
\end{aligned}$$

Remark 4.62. Suppose

$$f : X = \bigcup X_k \to Y = \bigcup Y_k$$

is a continuous map of cell complexes respecting the cell structure, i.e., f restricts to a (continuous) map $f_k : X_k \subset Y_k$. Then the f_k give rise to a map of chain complexes $f_*^{\mathrm{cell}} : C^{\mathrm{cell}}(X) \to C^{\mathrm{cell}}(Y)$ and the resulting morphism on cellular homology is compatible with the one on singular homology:

$$\begin{tikzcd}
\mathrm{H}_*(X) \arrow[r, "\cong"] \arrow[d, "f_*"] & \mathrm{H}_*^{\mathrm{cell}}(X) \arrow[d, "f_*^{\mathrm{cell}}"] \\
\mathrm{H}_*(Y) \arrow[r, "\cong"] & \mathrm{H}_*^{\mathrm{cell}}(Y).
\end{tikzcd}$$

This is true since all maps in the proof above are functorial.

Example 4.63. For $n \in \mathbf{Z}$, consider the map

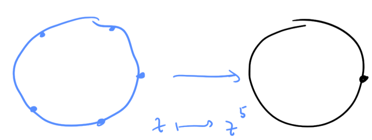
$$f : X := S^1 \to Y := S^1, z \mapsto z^n$$

(in complex number notation). We will show that the induced map

$$f_* := \mathrm{H}_1(f) : \mathrm{H}_1(X) \to \mathrm{H}_1(Y)$$

is multiplication by n. In other words, the mapping degree of f is n. This is clear for $n = 0$.

Next, we consider the case $n > 0$. We equip X and Y with different cell structures, namely X_0 has n points and $Y_0 = \{*\}$. Likewise, X_1 arises by glueing in n copies of Δ^1, while Y_1 only glues in one:



The map f is then compatible with the cell structure.

In order to show that f_* is multiplication by n, it suffices to see this for the map $\mathrm{H}_1^{\mathrm{cell}}(X) = \mathrm{H}_1(S^1, X_0) \to \mathrm{H}_1(S^1, Y_0)$. Suppose first that $n \geqslant 0$. We have an exact sequence

$$0 \to \underbrace{\mathrm{H}_1(S^1)}_{=\mathbf{Z}} \to \mathrm{H}_1(S^1, X_0) \to \underbrace{\mathrm{H}_0(X_0)}_{=\mathbf{Z}^n} \to \underbrace{\mathrm{H}_0(S^1)}_{=\mathbf{Z}} \to 0.$$

Writing $e_1, \dots, e_n$ for the 1-simplices in S^1 as shown, these are generators of $\mathrm{H}_1(S^1, X_0)$. Under f, they map to the loop in S^1 denoted e, which is in turn a generator of $\mathrm{H}_1(S^1, Y_0)$. In $\mathrm{H}_1(S^1)$, $e_1 + \cdots + e_n$ is a generator. By the above considerations, it maps to $e + \cdots + e = ne$.

If $n < 0$, the same argument still works, except that e_k maps to the loop e with its direction reversed. This reversed loop is, in $\mathrm{H}_1(S^1)$, the same as $-e$.

4.8 Homology with $\mathbf{Z}/2$-coefficients

In this section, we will consider homology with coefficients in $\Lambda = \mathbf{Z}/2$ in order to prove the following theorem, which can be stated colloquially by saying that at any moment in time there is a place x on earth such that wind and temperature at x and at its antipode $-x$ agree.

Theorem 4.64. (*Borsuk–Ulam theorem*) Let $f : S^n \to \mathbf{R}^n$ be a continuous map. Then there exists some $x \in S^n$ such that $f(x) = f(-x)$.

This theorem rests on the computation of homology of $\mathbf{RP}^n$ with $\mathbf{Z}/2$-coefficients: According to the computation of $\mathrm{H}_*(\mathbf{RP}^n)$ in Example 4.60 (and Theorem 4.61) and the short exact sequences (cf. Example 4.14)

$$0 \to \mathrm{H}_k(\mathbf{RP}^n)/2 \to \mathrm{H}_k(\mathbf{RP}^n, \mathbf{Z}/2) \to (\mathrm{H}_{k-1}(\mathbf{RP}^n))_2 \to 0,$$

we get

$$\mathrm{H}_k(\mathbf{RP}^n, \mathbf{Z}/2) = \begin{cases} \mathbf{Z}/2 & 0 \leqslant k \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{4.65}$$

and

$$\mathrm{H}_k(\mathbf{RP}^\infty, \mathbf{Z}/2) = \mathbf{Z}/2 \text{ for all } k \geqslant 0.$$

(By comparison, $\mathrm{H}_k(\mathbf{CP}^\infty) = \mathbf{Z}$ for all even $k \geqslant 0$, and 0 otherwise.)

Recall that a continuous map

$$p : E \to B$$

is called a *fiber bundle* if each point $b \in B$ admits an open neighborhood $U \ni b$ such that there is a homeomorphism fitting into a commutative diagram (for some topological space F):

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[dr, "p"'] & U \times F \arrow[d, "\mathrm{pr}"] \\ & U \end{tikzcd}$$

For simplicity, we will only apply this concept when B is connected, in which case all fibers (for all $b \in B$) are homeomorphic to one another. Then it makes sense to refer to F as "the" fiber (as opposed to "a" fiber; note that F is homeomorphic to $p^{-1}(b)$.) The space B is called the *base* and E the *total space*. If the fiber F is a discrete topological space, then p is called a *covering*.

Example 4.66.

- $\mathbf{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ is a covering (with fiber $\mathbf{Z}$).
- The canonical map $S^n \to \mathbf{RP}^n$ is a covering (with fiber $\mathbf{Z}/2$). We also refer to it as a *double covering*.

In order to compute the homology of covering spaces, we use the following fact from homotopy theory (for a proof see, e.g., [May99, §3.2]).

Proposition 4.67. Let $p : E \to B$ be a covering. For any n-simplex in $\mathrm{Sing}(B)$, i.e., $\sigma : \Delta^n \to B$, there is a *lift* of σ to E, i.e., a continuous map $\tilde{\sigma}$ making the diagram commutative:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ \Delta^n \arrow[ur, dotted, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & B. \end{tikzcd}$$

If the fiber F has n elements, then there are exactly n such maps $\tilde{\sigma}$.

Proposition 4.68. Let $p : E \to B$ be a double covering (i.e., $p^{-1}(b) = \{x, y\}$). Then there is a short exact sequence of chain complexes

$$0 \to C(B, \mathbf{Z}/2) \xrightarrow{\tau} C(E, \mathbf{Z}/2) \xrightarrow{p_*} C(B, \mathbf{Z}/2) \to 0,$$

where $\tau(\sigma) := \tilde{\sigma}_1 + \tilde{\sigma}_2$ and $p_*(\sigma) := p \circ \sigma$. Thus, the snake lemma (Lemma 4.13) gives long exact sequences

$$\ldots \to \mathrm{H}_n(B, \mathbf{Z}/2) \to \mathrm{H}_n(E, \mathbf{Z}/2) \to \mathrm{H}_n(B, \mathbf{Z}/2) \to \mathrm{H}_{n-1}(B, \mathbf{Z}/2) \to \ldots$$ ∎

Proof. First of all, we have $p_* \circ \tau = 0$, since $p_*\tilde{\sigma}_1 + p_*\tilde{\sigma}_2 = 2\sigma$, which vanishes in $C(B, \mathbf{Z}/2)$! It is then a routine check to show the exactness of the sequence. □

Corollary 4.69. Let $n \geqslant 1$ and let $f : S^n \to S^n$ be a continuous map such that $f(x) = -f(-x)$. Then f has odd mapping degree.

Proof. This proof is based on computations of homology with $\mathbf{Z}/2$-coefficients so we abbreviate $C(-) := C(-, \mathbf{Z}/2)$ and $\mathrm{H}_*(-) = \mathrm{H}_*(-, \mathbf{Z}/2)$. We have to show that f_* is an isomorphism on $\mathrm{H}_n(S^n)$.

By assumption on f, it induces a (continuous) map

$$\overline{f} : \mathbf{RP}^n \to \mathbf{RP}^n.$$

Since the two lifts $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ of a chain σ on S^n are antipodal, the following is a commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & C(\mathbf{RP}^n) \arrow[r, "\tau"] \arrow[d, "\overline{f}_*"] & C(S^n) \arrow[r, "p_*"] \arrow[d, "f_*"] & C(\mathbf{RP}^n) \arrow[r] \arrow[d, "\overline{f}_*"] & 0 \\
0 \arrow[r] & C(\mathbf{RP}^n) \arrow[r, "\tau"] & C(S^n) \arrow[r, "p_*"] & C(\mathbf{RP}^n) \arrow[r] & 0.
\end{tikzcd}$$

This yields a commutative diagram of long exact sequences (all homologies with $\mathbf{Z}/2$-coefficients), where we write $\overline{f}_n$ for $\overline{f}_*$ acting on the n-th homology etc.:

$$\begin{tikzcd}
\mathrm{H}_{k+1}(\mathbf{RP}^n) \arrow[r, "\delta"] \arrow[d, "\overline{f}_{k+1}"] & \mathrm{H}_k(\mathbf{RP}^n) \arrow[r, "\tau"] \arrow[d, "\overline{f}_k"] & \mathrm{H}_k(S^n) \arrow[r, "p_*"] \arrow[d, "f_k"] & \mathrm{H}_k(\mathbf{RP}^n) \arrow[d, "\overline{f}_k"] \\
\mathrm{H}_{k+1}(\mathbf{RP}^n) \arrow[r, "\delta"] & \mathrm{H}_k(\mathbf{RP}^n) \arrow[r, "\tau"] & \mathrm{H}_k(S^n) \arrow[r, "p_*"] & \mathrm{H}_k(\mathbf{RP}^n).
\end{tikzcd}$$

By the five lemma and Exercise 4.9, it therefore suffices to show that the maps $\overline{f}_k$ are isomorphisms for all k. This holds for $k \geqslant n+1$, since then $\mathrm{H}_k(\mathbf{RP}^n) = 0$ by the computation in (4.65). This vanishing also implies that $\tau : \mathrm{H}_n(\mathbf{RP}^n) \to \mathrm{H}_n(S^n)$ is an injective map of finite-dimensional $\mathbf{Z}/2$-vector spaces (of the same dimension), so it is an isomorphism. Thus, in high degrees, the sequences read

$$0 \to \mathrm{H}_n(\mathbf{RP}^n) \xrightarrow{\delta} \mathrm{H}_{n-1}(\mathbf{RP}^n) \to \mathrm{H}_{n-1}(S^n) = 0,$$

and $\overline{f}_n$ is (up to isomorphism) the same map as $\overline{f}_{n-1}$ etc. until we reach the end of the sequence where

$$\ldots \to \mathrm{H}_1(\mathbf{RP}^n) \to \mathrm{H}_0(\mathbf{RP}^n) \xrightarrow{\tau} \mathrm{H}_0(S^n) \xrightarrow{p_*} \mathrm{H}_0(\mathbf{RP}^n) \to 0.$$

Since both S^n and $\mathbf{RP}^n$ are connected, their H_0 with integral coefficients is $\mathbf{Z}$, hence $\mathrm{H}_0(-, \mathbf{Z}/2) = \mathbf{Z}/2$. Moreover, the map p_* is (isomorphic to) $\mathrm{id}_{\mathbf{Z}/2}$. Thus $\tau = 0$ and we again get $\overline{f}_1 = \overline{f}_0$. That last map $\overline{f}_0$ is, again, (isomorphic to) $\mathrm{id}_{\mathbf{Z}/2}$. Hence all $\overline{f}_n$ are isomorphisms, and hence f_n is an isomorphism as well. □

Proof. (of Theorem 4.64) We assume the contrary. Then our map $f : S^n \to \mathbf{R}^n$ yields a continuous *odd* map

$$g : S^n \to S^{n-1}, x \mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

The composite with the canonical inclusion $i : S^{n-1} \to S^n$ has $\deg(i \circ g) = 0$, since $\mathrm{H}_n(S^{n-1}) = 0$. On the other hand $g \circ i$, like g itself, is an odd map, so that $\deg(g \circ i)$ is odd by Corollary 4.69. However, its effect on homology is zero:

$$\mathbf{Z} = \mathrm{H}_{n-1}(S^{n-1}) \stackrel{i_*}{\to} \mathrm{H}_{n-1}(S^n) \stackrel{!}{=} 0 \stackrel{g_*}{\to} \mathrm{H}_{n-1}(S^{n-1}),$$

giving the required contradiction. □

4.9 Outlook: the Eilenberg–Steenrod axioms

We finish this chapter with an axiomatic point of view on singular homology. Let Pairs be the category whose objects are pairs (X, A) consisting of a topological space X and a subspace $A \subset X$, and whose morphisms are continuous maps $X \to X'$ such that A is mapped to A'.

In the sequel we consider (abstract) functors

$$h_n : \text{Pairs} \to \text{Ab},$$

i.e., an assignment $(X, A) \mapsto h_n(X, A) \in \text{Ab}$. Given such a functor, we write $h_n(X) := h_n(X, \varnothing)$. We have an obvious functor R : Pairs $\to$ Pairs, $(X, A) \mapsto (A, \varnothing)$.

Definition 4.70. A *generalized homology theory* is a collection of functors

$$h_n : \text{Pairs} \to \text{Ab}, n \geqslant 0$$

together with natural transformations (sometimes called *connecting homomorphisms*)

$$d_n : h_n \to h_{n-1} \circ R$$

such that the following conditions are satisfied:

(1) The functoriality of h_n and the natural transformation d_n, applied to the pair (X, A) constitute long exact sequences

$$\ldots \to h_n(A) \to h_n(X) \to h_n(X, A) \stackrel{d_n(X,A)}{\longrightarrow} h_{n-1}(A) \to \ldots$$

(2) (Additivity) There are (functorial) isomorphisms

$$\bigoplus_{i \in I} h_*(X_i) \xrightarrow{\cong} h_*(\bigsqcup X_i).$$

(3) (Homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic relative to A (i.e., the homotopy $h : \Delta^1 \times X \to Y$ maps $\Delta^1 \times A$ to B) then

$$h_*(f) = h_*(g) : h_*(X, A) \to h_*(Y, B).$$

(4) (Excision) If $\overline{Z} \subset A^\circ$ then the inclusions yield isomorphisms

$$h_*(X \backslash Z, A \backslash Z) \cong h_*(X, A).$$

Example 4.71. The contents of everything up to §4.5, except for the pretty obvious dimension axiom, can be summarized in on sentence: singular homology is a generalized homology theory.

Proposition 4.72. Suppose h_* and k_* are generalized homology theories, and

$$F : h_* \to k_*$$

is a natural transformation between them, i.e., the maps $F(X, A) : h_n(X, A) \to k_n(X, A)$ are functorial in the pair (X, A), and likewise the connecting homomorphisms:

$$\begin{array}{ccc} h_n(X, A) & \xrightarrow{F(X,A)} & k_n(X, A) \\ \downarrow & & \downarrow \\ h_{n-1}(A) & \xrightarrow{F(A,\varnothing)} & k_{n-1}(A). \end{array}$$

Suppose further that for a point $\mathrm{pt} := \{*\}$, we get an isomorphism

$$F(\mathrm{pt}) : h_*(\mathrm{pt}) \xrightarrow{\cong} k_*(\mathrm{pt}).$$

Then, F is an isomorphism for all pair (X, A) consisting of a cell complex X and a sub-complex A.

The motivation of stating this proposition is the fact that singular homology is, up to functorial isomorphisms, the *only* generalized homology theory that satisfies the dimension axiom

$$\mathrm{H}_n(\mathrm{pt}) = \begin{cases} \mathbf{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Outlook 4.73. Generalized homology theories *not* satisfying the dimension axiom are harder to construct, but highly interesting. The *stable homotopy groups* defined by

$$h_n(X) := \pi_n^s(X) := \operatorname{colim}_{r\to\infty} \pi_{r+n}(\Sigma^n X)$$

give rise to another generalized homology theory. Here $\Sigma := S^1 \wedge -$ is the *suspension functor* and π_{n+r} denotes the $(n+r)$-th homotopy group. The transition maps

$$\pi_r(X) \to \pi_{r+1}(S^1 \wedge X) \to \pi_{r+2}(S^2 \wedge X) \to \dots$$

are given by using that $\pi_r(X)$ consists of homotopy classes of maps $S^r \xrightarrow{f} X$, and then f maps to

$$S^{r+1} \cong S^1 \wedge S^r \xrightarrow{\mathrm{id}\wedge f} S^1 \wedge X.$$

The *Freudenthal suspension theorem* states that the maps

$$\pi_{r+n}(S^r) \to \pi_{r+n+1}(\Sigma S^r) = \pi_{r+n+1}(S^{n+1})$$

are isomorphisms for $r > n+1$, see, for example, [Swi02, Theorem 6.26]. This motivates the name *stable* homotopy groups:

$$\pi_n^s(S^0) := \pi_{r+n}(S^r) \text{ for } r > n.$$

For example, non-trivial computations show:

$$\pi_1(S^0) = 0 \to \pi_2(S^1) = 0 \to \pi_3(S^2) = \mathbf{Z} \to \pi_4(S^3) = \mathbf{Z}/2 \xrightarrow{\cong} \pi_5(S^4) \xrightarrow{\cong} \dots$$

so that $\pi_1^s(S^0) = \mathbf{Z}/2$. Understanding the stable homotopy groups of spheres is a matter of ongoing research. The Wikipedia article https://en.wikipedia.org/wiki/Homotopy_groups_of_spheres surveys the richness of this topic.

Proof. (of Proposition 4.72) We only sketch the main ideas, see [Swi02, Theorem 7.55] for complete details. By the five lemma, it suffices that the natural transformation induces isomorphisms when applied to cell complexes (as opposed to pairs consisting of such).

As in Exercise 4.11, one shows that the maps

$$h_n(X, \{x_0\}) \xleftarrow{\delta} h_{n+1}(CX, X) \to h_{n+1}(CX/X, \{*\}) = h_{n+1}(\Sigma X, \{*\})$$

are isomorphisms, where CX is the cone and ΣX the suspension of X. Since $S^n = \Sigma(\Sigma(\dots S^0))$ (n-fold suspension), we obtain isomorphisms $h_*(S^n) \cong k_*(S^n)$. The proof of Lemma 4.53 was based solely on the Eilenberg–Steenrod axioms, hence one has isomorphisms

$$\bigoplus_{i\in I} k_*(S^n, \{*\}) \xrightarrow{\cong} k_*(\bigvee_{i\in I} S^n, \{*\}).$$

If $X = \bigcup_{n=0}^m$ is a finite cell complex, then in order to show $h_*(X_n) \cong k_*(X_n)$, we proceed inductively, using that both for h_* and for k_*, we can compute the relative groups

$$h_*(X_n, X_{n-1}) = h_*(\bigvee_{j\in J_n} S^n)$$

which by the above agrees with the value for k_* instead of h_*.

For an infinite cell complex $X = \operatorname{colim} X_n$, one argues further, using the additivity again that $\operatorname{colim} h_*(X_n) = h_*(X)$, see [Swi02, Proposition 7.53]. $\square$

4.10 Exercises

Exercise 4.1. Using the stereographic projection, compute the homology groups of $S^n \backslash \{(1, 0, \dots, 0)\}$.

Exercise 4.2. Verify that HoTop is indeed a category and that there is an "obvious" functor

$$\mathrm{Top} \to \mathrm{HoTop}.$$

Exercise 4.3. Using Proposition 4.3 and Example 4.11, show that the homology of the following figure is isomorphic to the one of $\mathbf{R}^3 \backslash (\mathbf{R} \times (0,0) \sqcup \mathbf{R} \times (0,1))$. We will eventually show that these homology groups are given by $\mathbf{Z}$ in degrees 0 and 2, and $\mathbf{Z} \oplus \mathbf{Z}$ in degree 1.

Exercise 4.4. Let $A, B \subset Z$ be subsets of some set Z. Consider the obvious maps:

$$\begin{tikzcd} A \cap B \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & A \cup B. \end{tikzcd}$$

Show that there is a short exact sequence of abelian groups

$$0 \to \mathbf{Z}[A \cap B] \stackrel{g'+f'}{\to} \mathbf{Z}[B] \oplus \mathbf{Z}[A] \stackrel{f-g}{\to} \mathbf{Z}[A \cup B] \to 0.$$

Exercise 4.5. Let X be a simplicial set and $\ell \in \mathbf{Z}$. Using Example 4.14, prove that there is an exact sequence

$$0 \to \mathrm{H}_n(X)/\ell \to \mathrm{H}_n(X, \mathbf{Z}/\ell) \to (\mathrm{H}_{n-1}(X))_\ell \to 0$$

where again $(-)_\ell$ denotes the ℓ-torsion part of the group.

Tabulate these exact sequences for the simplicial spheres $X = S^k$ and for the projective plane $X = P^2$.

Exercise 4.6. Let A be a simplicial abelian group. The goal of this exercise is to prove the following fact: the natural map of chain complexes

$$p : C := C(A) \to N := N(A)$$

is a quasi-isomorphism.

(1) Let $D := D(A) \subset C(A)$ be the subcomplex which is in degree n given by A_n^{deg} (cf. Definition and Lemma 3.9). Show that the above claim is equivalent to the assertion that the complex D is exact.

Hint: show that $0 \to D \to C \to N \to 0$ is an exact sequence of complexes.

(2) Let C be a chain complex and $C' \subset C$ a subcomplex (i.e., $C'_n \subset C_n$ and $\partial_{C'} = \partial_C$). Show: if C' and the quotient complex C/C' are exact, then C is exact.

(3) For $p \geqslant 0$, let $D^{(p)} \subset D$ be defined by

$$(D^{(p)})_n := \begin{cases} D_n & n \leqslant p \\ \sigma_0(C_{n-1}) + \cdots + \sigma_p(C_{n-1}) & n > p \end{cases}$$

Show this defines a sequence of subcomplexes

$$\cdots \subset D^{(p-1)} \subset D^{(p)} \subset \cdots \subset D (\subset C).$$

(4) Show that the quotients $D^{(p)}/D^{(p-1)}$ are *null-homotopic*, i.e., there is a homotopy between the identity map of this complex and the zero map.

(5) Conclude that D is exact.

Exercise 4.7. Let $X := \overline{B}(R, x) \in \mathbf{R}^n$ be a *closed ball* with radius $R > 0$. Let $y \in X$ be an arbitrary point. Compute $\mathrm{H}_n(X, X \backslash \{y\})$. (Hint: the result depends on whether y is a point on the boundary of X or not).

Exercise 4.8. Let X and Y be two topological manifolds. Show that X and Y have the same dimension if they are homeomorphic. The converse is false: prove that there exist connected surfaces (i.e., manifolds of dimension two) which are not homeomorphic.

Exercise 4.9. "Long exact sequences can be broken into short exact sequences." More formally, let

$$\dots C_{n+2} \stackrel{\partial_{n+2}}{\to} C_{n+1} \stackrel{\partial_{n+1}}{\to} C_n \stackrel{\partial_n}{\to} C_{n-1} \stackrel{\partial_{n-1}}{\to} C_{n-2} \stackrel{\partial_{n-2}}{\to} \dots$$

be a long exact sequence. Construct a short exact sequence

$$0 \to \mathrm{coker}(\partial_{n+2}) \to C_n \to \ker(\partial_{n-1}) \to 0.$$

Exercise 4.10. The *Euler characteristic* of a topological space X is defined as

$$\chi(X) := \sum_{k \geqslant 0} (-1)^k \operatorname{rk} \mathrm{H}_k(X) = \sum_k (-1)^k \dim_{\mathbf{Q}} \mathrm{H}_k(X, \mathbf{Q}),$$

provided that only finitely many $\mathrm{H}_k(X)$ have non-zero rank and that all these ranks are finite. The rank $\operatorname{rk} \mathrm{H}_k(X)$ is called the k-th *Betti number*.

(1) In the situation of Corollary 4.20 suppose that $\chi(U)$, $\chi(V)$ and $\chi(U \cap V)$ are defined. Show that $\chi(X)$ is defined and that

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Hint: use Exercise 4.9 and find out what the rank–nullity theorem tells you about the ranks of the groups in a short exact sequence $0 \to A \to B \to C \to 0$.

(2) Prove that for a cell complex $X = \bigcup_k X_k$,

$$\chi(X) = \sum (-1)^k n_k,$$

where n_k is the number of cells glued in (where we suppose $n_k = 0$ for $k \gg 0$ and all n_k are finite).

(3) Compute the Euler characteristics of S^4, $S^3 \times S^1$, $S^2 \times S^2$, and $S^1 \times S^1 \times S^1 \times S^1$, and $\mathbf{CP}^2$.

Hint: both (1) or (2) can be used for such computations.

(4) Just using the Euler characteristic, which of these five spaces can you prove to be not homeomorphic?

(5) (Optional, bonus) Compute the Betti numbers of these spaces and conclude that, in fact, none of these spaces are pairwise homeomorphic.

Exercise 4.11. The *cone* CX of a topological space X is defined as

$$CX := X \times [0,1] \sqcup_{X \times \{0\}} \{0\},$$

while the *suspension* is defined as

$$\Sigma X := X \times [0,1] \sqcup_{X \times \{0,1\}} \{0,1\} (= CX/X \times \{1\}).$$

- Show that S^n is homeomorphic to ΣS^{n-1} (including for $n = 0$ if we put $S^{-1} := \varnothing$).
- Construct a natural isomorphism

$$\widetilde{\mathrm{H}}_n(\Sigma X) \cong \widetilde{\mathrm{H}}_{n-1}(X).$$

  Hint: inspect the Mayer–Vietoris sequence for an appropriate covering of ΣX.
- Reprove the computation of the homology of S^n.

Exercise 4.12. [Hat02, §2, Exercise 7] Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be an invertible linear map. Show that the induced map on the local homology group

$$f_* : \mathrm{H}_n(\mathbf{R}^n, \mathbf{R}^n \backslash \{0\}) \to \mathrm{H}_n(\mathbf{R}^n, \mathbf{R}^n \backslash \{0\})$$

equals multiplication by $\operatorname{sgn}\det(f) (\in \{+1, -1\})$.

Hint: Use Gaussian elimination to show that the matrix of f can be joined by a path of invertible matrices to a diagonal matrix with ± 1's on the diagonal.

Chapter 5

Singular cohomology

Singular cohomology is another invariant of simplicial sets and topological spaces. The cohomology groups are denoted by

$$\mathrm{H}^n(X).$$

On the face of it, it just arises by essentially reversing (or, rather, dualizing) the arrows in the normalized chain complexes. Therefore it is closely related to and, in several cases, even agrees with, homology. The advantage of cohomology is that there are maps, called *cup products*

$$\mathrm{H}^n(X) \times \mathrm{H}^m(X) \to \mathrm{H}^{n+m}(X),$$

which is a feature that homology groups do not have. These cup products can be used to equip the direct sum $\bigoplus_n \mathrm{H}^n(X)$ with the structure of a commutative ring. For example, for $X = \mathbf{CP}^2$, we already know $\mathrm{H}_n(\mathbf{CP}^2) = \mathbf{Z}$ for $n = 0, 2, 4$. We will prove below (Theorem 5.33) that the multiplication is such that the generator $\omega \in \mathrm{H}^2(\mathbf{CP}^2)$ generates the ring, so that there is a ring isomorphism:

$$\bigoplus_n \mathrm{H}^n(\mathbf{CP}^3) = \mathbf{Z}[\omega]/\omega^3.$$

5.1 Definition and examples

Cohomology arises by homology by dualizing. For an abelian group M, we write

$$M^\vee := \mathrm{Hom}(M, \mathbf{Z})$$

for the *dual abelian group*. We will mostly apply this to M being a free abelian group, in which case we have

$$\left(\bigoplus_{i \in I} \mathbf{Z}\right)^\vee = \prod_{i \in I} \mathbf{Z}.$$

(For countably infinite I, this is known to be a non-free abelian group.)

Any group homomorphism $f : M \to N$ gives rise to a map

$$f^\vee : N^\vee \to M^\vee, (N \xrightarrow{n} \mathbf{Z}) \mapsto (M \xrightarrow{f} N \xrightarrow{n} \mathbf{Z}).$$

This constitutes a functor

$$-^\vee : \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}.$$

This construction extends to a functor taking values in the category of *cochains*

$$\begin{aligned} -^\vee : \mathrm{Ch}^{\mathrm{op}} &\to \mathrm{CoCh}, \\ C &\mapsto C^\vee := [\dots \to C_{n-1}^\vee \xrightarrow{(\partial_n^C)^\vee} (C_n)^\vee \xrightarrow{(\partial_{n+1}^C)^\vee} (C_{n+1})^\vee \to \dots.] \end{aligned}$$

Indeed, the composite vanishes:

$$(\partial_{n+1}^C)^\vee \circ (\partial_n^C)^\vee = (\partial_n^C \circ \partial_{n+1}^C)^\vee = 0^\vee = 0.$$

Here, we regard $C_n^\vee$ to be in cochain degree $+n$ (so that the differential goes up by $+1$).

Definition 5.1. The n-th *cohomology functor* is the following composite:

$$\mathrm{sSet}^{\mathrm{op}} \xrightarrow{\mathbf{Z}[-]} \mathrm{sAb}^{\mathrm{op}} \xrightarrow{N} \mathrm{Ch}^{\mathrm{op}} \xrightarrow{-^\vee} \mathrm{CoCh} \xrightarrow{\mathrm{H}^n} \mathrm{Ab}.$$

More concretely, for a simplicial set X,

$$\mathrm{H}^n(X)$$

is the cohomology (at the spot $N(X)_n^\vee$) of the cochain complex

$$\dots \to (N(X)_{n-1})^\vee \xrightarrow{(\partial_n)^\vee} (N(X)_n)^\vee \xrightarrow{(\partial_{n+1})^\vee} (N(X)_{n+1})^\vee \to \dots.$$

The *singular cohomology* of a topological space X is defined as

$$\mathrm{H}^n(X) := \mathrm{H}^n(\mathrm{Sing}(X)).$$

Thus, cohomology is a functor

$$\mathrm{H}^n : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Ab}.$$

Again, for a commutative ring Λ, we define cohomology with coefficients in Λ by replacing the free abelian group functor $\mathbf{Z}[-]$ by $\Lambda[-]$. We denote the result by

$$\mathrm{H}^n(X, \Lambda).$$

We first look at cohomologies of a few simplicial sets.

Example 5.2. For $k \geqslant 1$, the dual of the chain complex $N(S^k) = [\mathbf{Z} \xrightarrow{0} 0 \ldots \to 0 \to \mathbf{Z}]$ (with the left hand $\mathbf{Z}$ in degree k, the right one in degree 0) is the cochain complex

$$N(S^k)^\vee = [\mathbf{Z} \xleftarrow{0} 0 \cdots \leftarrow 0 \leftarrow \mathbf{Z}].$$

Since all differentials are (still) zero, we have

$$\mathrm{H}^n(S^k) = \mathrm{H}_n(N(S^k)^\vee) = \begin{cases} \mathbf{Z} & n = 0, k \\ 0 & \text{otherwise} \end{cases}$$

Thus, $\mathrm{H}^n(S^k) \cong (\mathrm{H}_n(S^k))^\vee$. This example is somewhat prototypical. More precisely, one can show:

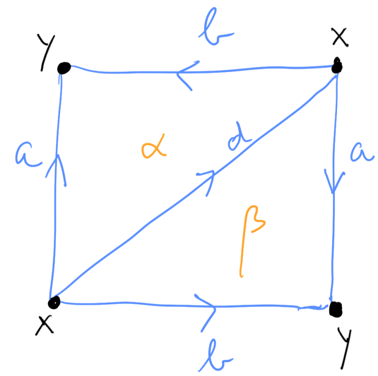
Proposition 5.3. Let X be a simplicial set or a topological space. Then

$$\mathrm{H}^n(X, \mathbf{Q}) = \mathrm{Hom}_{\mathbf{Q}}(\mathrm{H}_n(X, \mathbf{Q}), \mathbf{Q}),$$

i.e., cohomology *with rational coefficients* is just the homology, dualized.

With torsion coefficients, however, a more subtle relationship holds, as we see for the projective plane P^2, cf. Example 3.20.

Example 5.4. Recall that the (simplicial) projective plane P^2 is the simplicial set pictured as follows:



We had computed the normalized chain complex as

$$\Lambda_\alpha \oplus \Lambda_\beta \xrightarrow{\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}} \Lambda_a \oplus \Lambda_b \oplus \Lambda_d \xrightarrow{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}} \Lambda_x \oplus \Lambda_y.$$

We now pass to duals, writing $\Lambda^\alpha := (\Lambda_\alpha)^\vee$. With respect the the obvious dual bases, the dual then is the cochain complex

$$\Lambda^\alpha \oplus \Lambda^\beta \xleftarrow{\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}} \Lambda^a \oplus \Lambda^b \oplus \Lambda^d \xleftarrow{\begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}} \Lambda^x \oplus \Lambda^y.$$

For example, the basis vector e^x maps to $-e^a - e^b$ etc. We write Z^k, B^k for the cocycles and coboundaries of that complex. We see that

- $\mathrm{H}^0(P^2, \Lambda) = Z^0 = \{(x, x), x \in \Lambda\} \cong \Lambda$,
- $B^1 = \{(x, x, 0), x \in \Lambda\}$,
- $Z^1 = \{(x_a, x_b, x_d) | -x_a + x_b + x_d = 0, x_a - x_b + x_d = 0\} = \{(x_b + x_d), x_b, x_d, 2x_d = 0\} \cong \Lambda \oplus \Lambda_2$, where Λ_2 is again the 2-torsion subgroup of Λ.
- Thus the map

  $$Z^1/B^1 \to \Lambda_2, \{(x_b + x_d), x_b, x_d, 2x_d = 0\} \mapsto x_d$$

  yields an isomorphism

  $$\mathrm{H}^1(P^2, \Lambda) = \Lambda_2.$$

Indeed, the map is clearly surjective and it is injective since for $x_d = 0$ the triple lies in B_1.

- Finally $B^2 = \{(x_a + 2x_d, -x_a), x_a, x_d \in \Lambda\} \subset Z^2 = \Lambda^\alpha \oplus \Lambda^\beta$.
- The map

$$\mathrm{H}^2 = Z^2/B^2, (x_\alpha, x_\beta) \mapsto [x_\alpha + x_\beta] \in \Lambda/2$$

is an isomorphism: it is clearly surjective and also injective since if $x_\alpha + x_\beta = 2x$ for some $x \in \Lambda$, the pair is equal to $(2x - x_\beta, x_\beta) \in B^2$.

We sum up our findings, and notice a more subtle relationship between homology and cohomology than in the case of the k-sphere:

$$\mathrm{H}^n(P^2) = \begin{cases} \Lambda & n = 0 \\ \Lambda_2 & n = 1 \\ \Lambda/2 & n = 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{compared to } \mathrm{H}_n(P^2) = \begin{cases} \Lambda & n = 0 \\ \Lambda/2 & n = 1 \\ \Lambda_2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Remark 5.5. The observation that in the above computation Λ_2 gets exchanged by $\Lambda/2$ can be explained as follows: the complex (in degrees 1 and 0 as labelled)

$$\Lambda_1 \xrightarrow{2} \Lambda_0$$

(2 stands for the map given by multiplication by 2) has homology $\mathrm{H}_1 = \ker 2 = \Lambda_2$ and $\mathrm{H}_0 = \operatorname{coker} 2 = \Lambda/2$. Passing to duals, we get the cochain complex ($\Lambda^i := \Lambda_i^\vee$ lives in cochain degree i)

$$\Lambda^0 \xrightarrow{2} \Lambda^1$$

which now has $\mathrm{H}^0 = \ker 2 = \Lambda_2$ and $\mathrm{H}^1 = \operatorname{coker} 2 = \Lambda/2$.

Such an observation is at the heart of the so-called *universal coefficient theorem for cohomology* which expresses $\mathrm{H}^n(X)$ in terms of $\mathrm{Hom}(\mathrm{H}_n(X), \mathbf{Z})$ and a so-called Ext-group $\mathrm{Ext}(\mathrm{H}_{n-1}(X), \mathbf{Z})$. See, e.g., [Rot88, Theorem 12.11].

This second group vanishes whenever $\mathrm{H}_{n-1}(X)$ is a free $\mathbf{Z}$-module. Thus, in this case we get an isomorphism

$$\mathrm{H}^n(X) = \mathrm{Hom}(\mathrm{H}_n(X), \mathbf{Z}).$$

5.2 The Eilenberg–Steenrod axioms for cohomology

Singular cohomology satisfies the following properties. This theorem can be proven by redoing the proofs for homology.

Theorem 5.6. (1) (Functoriality) For each $n \geqslant 0$, there is a functor

$$\mathrm{H}^n : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Ab}.$$

(2) (Dimension axiom) The groups $\mathrm{H}^n(\{*\})$ are zero for $n \neq 0$, and H_0 is isomorphic to $\mathbf{Z}$.

(3) (Additivity) For a family of topological spaces (X_i), we have

$$\mathrm{H}^n(\bigsqcup_i X_i) = \prod_i \mathrm{H}^n(X_i).$$

(4) (Homotopy) Homotopic maps $f, g : X \to Y$ induce the same map on cohomology:

$$\mathrm{H}^n(f) = \mathrm{H}^n(g).$$

In particular, homotopy equivalences induce isomorphisms on cohomology.

(5) (Mayer–Vietoris sequence) If X is a topological space, $U, V \subset X$ such that their interiors cover X: $X = U^\circ \cup V^\circ$, then there is a long exact sequence

$$\ldots \to \mathrm{H}^n(X) \to \mathrm{H}^n(U) \oplus \mathrm{H}^n(V) \to \mathrm{H}^n(U \cap V) \to \mathrm{H}^{n+1}(X) \to \ldots.$$

Example 5.7.

$$\mathrm{H}^n(S^k) = \begin{cases} \mathbf{Z} & n = 0, k \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{H}^n(\mathbf{CP}^k) = \begin{cases} \mathbf{Z} & n = 0, 2, 4, \ldots, 2k \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{H}^n(\mathbf{RP}^k, \mathbf{Z}/2) = \begin{cases} \mathbf{Z}/2 & n = 0, 1, \ldots, k \\ 0 & \text{otherwise} \end{cases}$$

These computations can be confirmed by using the above axioms, in particular the Mayer–Vietoris sequence. Alternatively they also follow from Remark 5.5.

5.3 The cup product

A key advantage of passing from homology to cohomology is the existence of cup products on the latter. These maps, constructed soon, are of the form

$$\cup : \mathrm{H}^n(X) \otimes \mathrm{H}^m(X) \to \mathrm{H}^{n+m}(X).$$

This ring structure is a finer structure than the mere existence of the abelian groups $\mathrm{H}_n(X)$ or $\mathrm{H}^n(X)$.

Example 5.8. Let $X = S^2 \vee S^1 \vee S^1$ and $Y = T = S^1 \times S^1$ be the torus. These two spaces are both path-connected, so $\mathrm{H}_0(X) = \mathrm{H}_0(Y) = \mathbf{Z}$. For $n \geqslant 1$, by using additivity (of reduced homology, Lemma 4.53), we have

$$\mathrm{H}_n(X) = \mathrm{H}_n(S^2) \oplus \mathrm{H}_n(S^1) \oplus \mathrm{H}_n(S^1) = \begin{cases} \mathbf{Z} & n = 2 \\ \mathbf{Z} \oplus \mathbf{Z} & \mathrm{n} = 1 \\ 0 & n \geqslant 3 \end{cases}$$

These groups are isomorphic to

$$\mathrm{H}_n(T) = \mathrm{H}_n^{\mathrm{cell}}(T) = \mathrm{H}_n(\mathbf{Z} \xrightarrow{0} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{0} \mathbf{Z}).$$

So, homology *groups* and therefore also the cohomology *groups* are isomorphic.

However, it turns out that the cup product of the two generators in $\mathrm{H}^1(X)$ *vanish* (this is always the case for wedge sums of spaces). By contrast, we will momentarily see that the two loops in T which are the two generators of $\mathrm{H}^1(T)$ have the property that

$$\gamma_1 \cup \gamma_2 \in \mathrm{H}^2(T)$$

is non-zero, and instead in fact the generator of $\mathrm{H}^2(T) \cong \mathbf{Z}$. Thus, there can be no homotopy equivalence

$$f : X \to Y$$

since out would have to induce (as we will see) an isomorphism of *rings*

$$\mathrm{H}^*(f) : \bigoplus_{n \geqslant 0} \mathrm{H}^n(X) \to \bigoplus_{n \geqslant 0} \mathrm{H}^n(Y).$$

Notation 5.9. For some $\sigma : \Delta^n \to X$ and a subset $J \subset [n]$, we write $\Delta^J \subset \Delta^n$ for the subset consisting of those $(x_0, \ldots, x_n)$ where $x_j = 0$ for $j \notin J$. We also write $\sigma|_J$ for the composite $\Delta^J \subset \Delta^n \xrightarrow{\sigma} X$.

Definition 5.10. Let X be a topological space. The *cup product* is the map

$$- \cup - : C^*(X) \times C^*(X) \to C^*(X)$$

which, for $a \in C^k(X)$ and $b \in C^l(X)$ is given by

$$(a \cup b)(\sigma) = \underbrace{a(\sigma|_{[0,k]})}_{\in \mathbf{Z}} \cdot \underbrace{b(\sigma|_{[k,k+l]})}_{\in \mathbf{Z}}.$$

Here $\sigma : \Delta^{k+l} \to X$. We extend this by bilinearity to a map

$$- \cup - : C^*(X) \otimes C^*(X) \to C^*(X).$$

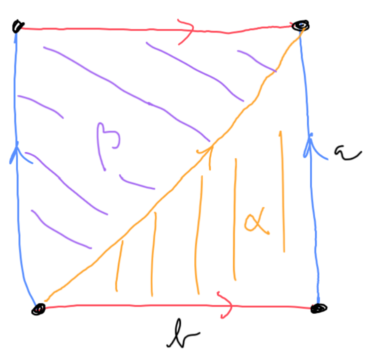
The same definition applies for cochains taking values in a commutative ring Λ.

Example 5.11. Let $X = T = S^1 \times S^1$ be the torus. The only interesting cup product is the map

$$\cup : \mathrm{H}^1(T) \otimes \mathrm{H}^1(T) \to \mathrm{H}^2(T).$$

(We will see below cup product with $n \in \mathrm{H}^0(T) = \mathbf{Z}$ is just multiplication by n, and if the degrees add up to $\geqslant 3$, the cup product must vanish since then $\mathrm{H}^n(T) = 0$.)

We consider the following vertices, 1-simplices and 2-simplices on T:



The dual cochains are denoted by $a^\vee$ etc., i.e., $a^\vee \in C^1(T)$ is the cochain satisfying

$$a^\vee(a) = 1, a^\vee(b) = 0, a^\vee(d) = 0.$$

We compute the cochain $a^\vee \cup b^\vee$ by applying the definition:

- $(a^\vee \cup b^\vee)(\alpha) = a^\vee(\alpha|_{[0,1]})b^\vee(\alpha|_{[1,2]}) = a^\vee(b) \cdot b^\vee(a) = 0,$
- $(a^\vee \cup b^\vee)(\beta) = a^\vee(\beta|_{[0,1]})b^\vee(\beta|_{[1,2]}) = a^\vee(a) \cdot b^\vee(b) = 1.$
- Thus, by linearity

$$(a^\vee \cup b^\vee)(\beta - \alpha) = 1.$$

- The 2-chain $\gamma := \beta - \alpha$ is a generator of $\mathrm{H}_2(T)$. So that its dual cochain $\gamma^\vee$ is a generator of $\mathrm{H}^2(T) = \mathrm{Hom}(\mathrm{H}_2(T), \mathbf{Z})$.
- We obtain

$$a^\vee \cup b^\vee = \gamma^\vee.$$

- A similar computation (!)shows

$$b^\vee \cup a^\vee = -\gamma^\vee.$$

- Again by the above computations, we have

$$a^\vee \cup a^\vee = 0, b^\vee \cup b^\vee = 0.$$

The above computation suggests the following statement: there is a ring isomorphism (where the right hand carries the cup product)

$$\mathbf{Z}\langle s, t\rangle/(st + ts, s^2, t^2) \xrightarrow{\cong} \bigoplus_{n=0}^{2} \mathrm{H}^n(T),$$

sending $s \mapsto a^\vee$, $t \mapsto b^\vee$ and $st \mapsto \gamma^\vee$. Here at the left $\mathbf{Z}\langle s, t\rangle$ denotes the *non-commutative polynomial ring* (which has as a $\mathbf{Z}$-basis the $1, s, t, st, ts, s^2, t^2, \dots$), and we mod out the 2-sided ideal generated by $st + ts$, s^2 and t^2. In order to make this statement, we need to exhibit how the cup product turns the sum $\bigoplus_n \mathrm{H}^n(X)$ into a ring. This ring structure comes in fact from a "ring" structure on $C^*(X)$ which is "essentially commutative."

Definition 5.12. A *differential graded algebra* (*dga* for short) is a cochain complex A together with a cochain map (called the multiplication)

$$\mu : A \otimes A \to A$$

(denoted by juxtaposition) such that the usual conditions on a commutative ring hold:

- (Unitality) There is an element $1 \in Z^0(A) = \mathrm{Hom}_{\mathrm{CoCh}}(\mathbf{Z}, A)$ such that $1a = a1 = a$.
- (Associativity) $a(bc) = (ab)c$.

A morphism of dga's is a cochain map $A \to B$ compatible with the multiplication maps in the obvious sense.

Lemma 5.13. The cup product turns $C^*(X)$ into a dga. For any continuous map $f : X \to Y$ the induced map

$$f^* : C^*(Y) \to C^*(X)$$

is compatible with the multiplications and the units. We refer to this by saying that f^* is a map of dga's. We have therefore a functor taking values in the category of dga's:

$$C^* : \mathrm{Top}^{\mathrm{op}} \to \mathrm{DGA}.$$

Proof. We first check that $\cup$ is a chain map. By definition of ∂ on tensor products of chain complexes, this means

$$\partial(a \cup b) = (\partial a) \cup b + (-1)^k a \cup (\partial b),$$

where again $a \in C^k(X)$. Recall that the differential ∂ on cochains is obtained by dualizing the differential on chains. Thus, for a $\sigma : \Delta^{k+l+1} \to X$,

$$\begin{aligned}
(\partial(a \cup b))(\sigma) &= (a \cup b)(\sum_{i=0}^{k+l+1} (-1)^i \sigma \circ \delta_i) \\
&= (a \cup b)(\sum_{i=0}^{k+l+1} (-1)^i \sigma|_{[0,\dots,\hat{\imath},\dots,k+l+1]}) \\
&= \sum_{i \leqslant k+1} (-1)^i a(\sigma|_{[0,\dots,\hat{\imath},\dots,k+1]}) \cdot b(\sigma|_{[k+1,k+l+1]}) + \sum_{i \geqslant k} a(\sigma|_{[0,k]}) \cdot b(\sigma_{[k,\dots,\hat{\imath},k+l+1]} \\
&= ((\partial a) \cup b)(\sigma) + (-1)^k (a \cup (\partial b))(\sigma).
\end{aligned}$$

The unit element is the cocycle 1 such that $1(\sigma) = 1$ for each 0-simplex σ in X. The associativity is a routine check, as is the compatibility of f^* with unit and multiplication maps. □

Lemma 5.14. Let C be a dga, and consider its cohomology

$$\mathrm{H}^*(C) := \bigoplus_{n \in \mathbf{Z}} \mathrm{H}^n(C).$$

The multiplication

$$[a] \cdot [b] := [ab]$$

turns this into a graded algebra, i.e., the same conditions as for a dga above hold (except $\mathrm{H}^*(C)$ carries no differential).

Proof. The multiplication $\mu : C \otimes C \to C$ is a cochain morphism so that

$$\partial(rs) := \partial\mu(r \otimes s) = \mu(\partial(r \otimes s)) = (\partial r)s + (-1)^{\deg r} r(\partial s).$$

Thus, if r and s are cocycles (i.e., map to 0 under ∂), the same is true for rs. In order to check this induces a multiplication on $\mathrm{H}^*(C)$, it suffices to check that $(\partial r)s$ and $r(\partial s)$ are coboundaries. Indeed, again by the above formula

$$(\partial r)s = \partial(rs) - (-1)^{\deg r} r(\partial s) = \partial(rs)$$

(for any cocycle s) is a coboundary. □

Combining the above, we see that cohomology is a functor

$$\mathrm{H} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{GA}$$

(taking values in the category of graded algebras). In fact, we can do better:

Proposition 5.15. The cup product on $\bigoplus_n \mathrm{H}^n(X)$ is in fact a *graded commutative ring*, i.e., we have

$$ab = (-1)^{\deg m \deg n} ba$$

for $a \in \mathrm{H}^m(X)$, $b \in \mathrm{H}^n(X)$.

Proof. (Proof idea) This graded commutativity comes from the following idea: put $\epsilon_n := (-1)^{n(n+1)/2}$ and define a map on the chain complexes

$$\rho : C_n(X) \to C_n(X), (\sigma : \Delta^n \to X) \mapsto \epsilon_n \cdot \sigma^{\mathrm{op}}.$$

Here

$$\sigma^{\mathrm{op}} : \Delta^n \to \Delta^n \xrightarrow{\sigma} X,$$

where the first map is the unique affine-linear map sending the basis vector e_k to e_{n-k}. One checks that this map ρ is a chain map. One also checks that there is a homotopy between ρ and the identity map.

With these claims checked, a quick algebraic manipulation implies that the multiplication on $\bigoplus_n \mathrm{H}^n(X)$ is graded commutative as stated. See, e.g., [Hat02, Theorem 3.14]. □

Example 5.16. For $X = S^1 \times S^1$ as in Example 5.11, a^{op} is the loop a, but with its direction reversed. Thus $\rho(a) = -a^{\mathrm{reversed}} (\in C_1(X))$. The asserted homotopy between ρ and id reduces to the fact that $a + a^{\mathrm{reversed}}$ is homotopic to a constant loop.

5.4 Poincaré duality

In this section, we explore Poincaré duality, one of the foundational results of algebraic topology. The idea is a certain symmetry between homology (or, cohomology) groups.

Example 5.17. $\mathrm{H}_k(S^n) = \mathrm{H}^k(S^n) = \mathbf{Z}$ in degrees 0 and n. The group vanishes otherwise. This can be recast as a symmetry

$$\begin{aligned} \mathrm{H}_k(S^n) &= \mathrm{H}_{n-k}(S^n)^\vee, \\ \mathrm{H}^k(S^n) &= \mathrm{H}^{n-k}(S^n)^\vee \end{aligned} .$$

(At this point, the appearance of the dual is unmotivated; we might as well write $\mathrm{H}_k(S^n) = \mathrm{H}_{n-k}(S^n)$, but see Theorem 5.30.)

Example 5.18. The space $X = S^1 \times S^3$ is a cell complex with one cell in dimension 0, 1, 3 and 4, respectively. One can compute (for example using Mayer–Vietoris)

$$\mathrm{H}_k(X) = \mathrm{H}^k(X) = \mathbf{Z}$$

for $k = 0, 1, 3, 4$, and the groups vanish in all other degrees. Thus, again

$$\begin{aligned} \mathrm{H}_k(X) &= \mathrm{H}_{4-k}(X)^\vee \\ \mathrm{H}^k(X) &= \mathrm{H}^{4-k}(X)^\vee. \end{aligned}$$

Example 5.19. For complex projective space

$$\mathrm{H}_k(\mathbf{CP}^n) = \mathrm{H}_{2n-k}(\mathbf{CP}^n)^\vee$$

and again likewise for cohomology.

Example 5.20. Consider $X = S^1 \vee S^1$. By additivity (for reduced homology), we get

$$\mathrm{H}_0(X) = \mathbf{Z}, \mathrm{H}_1(X) = \mathbf{Z} \oplus \mathbf{Z}.$$

Thus, the symmetry encountered above breaks:

$$\mathrm{H}_0(X) \neq \mathrm{H}_1(X)^\vee.$$

It is suggestive to link this behaviour to the presence of the "singular" point. In order to rule out this kind of pathology, we consider (topological) *manifolds*, i.e., topological spaces X such that each $x \in X$ has a neighborhood that is homeomorphic to an open ball in some $\mathbf{R}^d$, where d is independent of x. All the spaces above, except for $S^1 \vee S^1$ are manifolds, leading to the idea that for such a manifold X we might expect a close relation between

$$\begin{aligned} &\mathrm{H}_k(X) \text{ and } \mathrm{H}_{d-k}(X) \\ &\mathrm{H}^k(X) \text{ and } \mathrm{H}^{d-k}(X). \end{aligned}$$

Example 5.21. Real projective space

$$\mathbf{RP}^n = S^n / x \sim -x$$

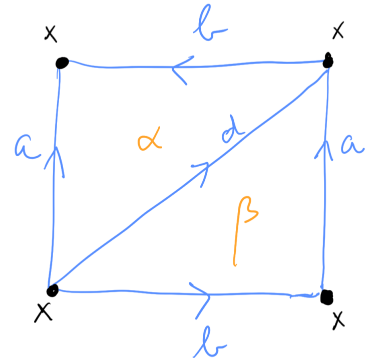
is a manifold of dimension n. However, the homology groups

$$\mathrm{H}_0(\mathbf{RP}^n) = 0,$$

$$\mathrm{H}_n(\mathbf{RP}^n) = \begin{cases} \mathbf{Z} & n \text{ odd} \\ \mathbf{Z}/2 & n \text{ even} \end{cases}$$

fail to be symmetric (say, when considering their ranks) when n is even.

Example 5.22. The *Klein bottle* X (cf. also Exercise 3.10) is the geometric realization of the following simplicial set:



It is a manifold of dimension 2. It is connected, so that $\mathrm{H}_0(X) = 0$. However, one can show $\mathrm{H}_2(X) = 0$. This can be done either using Mayer–Vietoris sequences or also using cellular homology. As a plausibility check, let us note that there are no 2-cycles for the above simplicial set:

$$\partial(n_\alpha \alpha + n_\beta \beta) = n_\alpha(b - a + d) + n_\beta(a - d + b) = a(n_\beta - n_\alpha) + b(n_\alpha + n_\beta) + d(n_\alpha - n_\beta) = 0$$

only if $n_\alpha = n_\beta = 0$. Thus, the above symmetry fails again.

To rule out this problem, we need to impose an extra condition on the manifolds we consider. We call a subset $B \subset X$ a (finite) open ball if there is a neighborhood $U \supset B$ that is homeomorphic to $\mathbf{R}^d$, under which B is homeomorphic to $B(0,1) \subset \mathbf{R}^d$.

For a subspace $A \subset X$, let us abbreviate

$$\mathrm{H}_*(X|A) := \mathrm{H}_*(X, X \backslash A).$$

For any manifold X and any open ball $B \subset X$, we have (compare with Example 4.35)

$$\mathrm{H}_d(X|B) = \mathrm{H}_d(U|B) \cong \mathrm{H}_d(\mathbf{R}^d|B(0,1)) = \mathrm{H}_d(S^d) \cong \mathbf{Z}.$$

The isomorphisms "=" are canonical. The left hand "≅" depends on the choice of a homeomorphism $U \cong \mathbf{R}^d$, and the right hand isomorphism can not be made canonical at all. Nonetheless, this tells us that the local homology group is a free abelian group of rank one. A choice of a generator $e_B \in \mathrm{H}_d(X|B)$ is called a *local orientation*. (I.e., there is always precisely two local orientations for each such B).

Remark 5.23. The above applies verbatim to $B = \{x\}$ as well, in which case we recover exactly Example 4.35.

Multiplication with a matrix $A \in \mathrm{GL}_d(\mathbf{R})$ is a homeomorphism

$$f : \mathbf{R}^d \to \mathbf{R}^d$$

respecting the subspace $\mathbf{R}^d \backslash \{0\}$. It therefore induces a map

$$f_* : \mathrm{H}_d(\mathbf{R}^d|0) \to \mathrm{H}_d(\mathbf{R}^d|0).$$

By Exercise 4.12, this map is given by multiplication with $\operatorname{sgn}(\det f) \in \{\pm 1\}$. Thus, an orthogonal matrix $A \in \mathrm{SL}_n(\mathbf{R})$ preserves the local orientation, while a map of the form $(x_1, \dots, x_d) \mapsto (-x_1, x_2, \dots, x_d)$ does not preserve it. This motivates the name "(local) orientation".

Definition 5.24. Let X be a manifold of dimension d. We call X *orientable* if one can choose local orientations compatibly for all open balls $B \subset X$. I.e., if there is a collection of generators

$$(e_B \in \mathrm{H}_d(X, X \backslash B))_{B \subset X \text{ open ball}}$$

such that for any inclusion of open balls $C \subset B$,

$$e_B \mapsto e_C$$

under the canonical map

$$\mathrm{H}_d(X|B) \to \mathrm{H}_d(X|C).$$

Example 5.25. $\mathbf{R}^d$ is orientable. To see this, fix a homeomorphism

$$e : \Delta^d \cong B(0, 1).$$

For $R > 0$, let $e_R : \Delta^d \xrightarrow{\cong} B(0, R)$ be the homeomorphism obtained by scaling. Now, for a bounded ball $B \subset \mathbf{R}^d$, consider the d-simplex, the d-simplex e_R is a generator of $\mathrm{H}_d(\mathbf{R}^d|B)$ as soon as $B \subset B(0, R)$. Note that $e_R = e_{R'}$ for $R' > R$. Thus, defining e_B to be e_R for R large enough, yields a compatible system of local orientations.

Definition 5.26. A *complex analytic manifold* X is a topological space where each $x \in X$ has a neighborhood $U \ni x$ that is homeomorphic to an open ball in $\mathbf{C}^d$:

$$f_U : U \xrightarrow{\cong} B(0, 1) \subset \mathbf{C}^d,$$

and where the transition maps

$$f_U \circ f_V^{-1}$$

are *complex differentiable* (also known as holomorphic), as opposed to a mere homeomorphism.

Example 5.27. Complex projective space $\mathbf{CP}^n$ is an example of a complex projective manifold, with open charts given by $(0 \leqslant i \leqslant n)$

$$\begin{aligned} f_i : U_i := \{(z_0, \dots, z_n), z_i \neq 0\}/\sim &\xrightarrow{\cong} \mathbf{C}^n, \\ z &\mapsto (\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots), \end{aligned}$$

where the equivalence relation is as in Example 4.46. Thus the transition functions are given by

$$(\frac{z_0}{z_i}, \dots, \frac{\widehat{z_i}}{z_i}, \dots, \frac{z_n}{z_i}) \mapsto (\frac{z_0}{z_j}, \dots, \frac{\widehat{z_j}}{z_j}, \dots, \frac{z_n}{z_j}).$$

This is essentially given by multiplication with $\frac{z_i}{z_j}$, which is a complex differentiable function. (In fact, since this is a quotient of polynomials, $\mathbf{CP}^n$ is an example of a complex *algebraic variety*).

Lemma 5.28. Any complex analytic manifold is orientable.

Proof. Proving this requires unwinding the definition of a complex manifold, but the basic point distinguishing real from complex manifolds is that a matrix $A \in \mathrm{GL}_d(\mathbf{C})$ has, when regarded as a matrix $\tilde{A} \in \mathrm{GL}_{2d}(\mathbf{R})$, *positive* determinant:

$$\det \tilde{A} = |\det A|^2 > 0.$$

This implies that the transition maps $f_U \circ f_V^{-1}$ will always preserve orientation. □

Example 5.29. Non-orientable manifolds include:

- $\mathbf{RP}^2$ (and all other *even-dimensional* real projective spaces),
- the Klein bottle,
- the Möbius strip.

It requires further means (such as proving Theorem 5.30) to rigorously show that these are non-orientable

Theorem 5.30. Let X be a *compact*, *orientable manifold* of dimension d. Then there is an isomorphism

$$\mathrm{H}^d(X) \cong \mathbf{Z}$$

and the cup product

$$\mathrm{H}^n(X) \otimes \mathrm{H}^{d-n}(X) \stackrel{\cup}{\to} \mathrm{H}^d(X) \cong \mathbf{Z}$$

is a perfect pairing, i.e., the induced map

$$\mathrm{H}^n(X) \to (\mathrm{H}^{d-n}(X))^\vee$$

is an isomorphism. In particular, the ranks of these groups, and therefore also of the corresponding homology groups agree:

$$\operatorname{rk} \mathrm{H}^n(X) = \operatorname{rk} \mathrm{H}^{d-n}(X), \operatorname{rk} \mathrm{H}_n(X) = \operatorname{rk} \mathrm{H}_{d-n}(X).$$

Remark 5.31. The three assumptions: compactness, orientability, and smoothness (i.e., being a manifold), can be removed at the expense of a more involved statement.

Question 5.32. Why does one have to assume X is compact for the above statement to be correct?

5.5 Cohomology of projective spaces

In this section, we compute the ring structure of cohomology on complex projective space $\mathbf{CP}^n$.

Theorem 5.33. There is a ring isomorphism

$$\mathbf{Z}[x]/x^{n+1} \stackrel{\cong}{\to} \mathrm{H}(\mathbf{CP}^n) := \bigoplus_{k \in \mathbf{Z}} \mathrm{H}^k(\mathbf{CP}^n). \tag{5.34}$$

Here the right hand side carries the cup product, and the element x in the left hand side has degree 2, i.e., it maps to an element in $\mathrm{H}^2(\mathbf{CP}^n)$.

Remark 5.35. This computation is one of a family of similar results such as

$$\mathbf{Z}[x] \cong \mathrm{H}(\mathbf{CP}^\infty),\ (\deg x = 2)$$
$$\mathbf{Z}/2[x]/x^{n+1} \cong \mathrm{H}(\mathbf{RP}^n, \mathbf{Z}/2),\ (\deg x = 1)$$
$$\mathbf{Z}/2[x] \cong \mathrm{H}(\mathbf{RP}^\infty, \mathbf{Z}/2),\ (\deg x = 1).$$

See, e.g., [Mas91, §XV].

By the presentation of $\mathbf{CP}^n$ as a cell complex with one cell in every *even* dimension, we have

$$\mathrm{H}_k(\mathbf{CP}^n) = \mathrm{H}^{\mathrm{cell}}_k(\mathbf{CP}^n) = \mathbf{Z}$$

whenever $0 \leqslant k \leqslant 2n$ is even, and the groups vanish otherwise. By Remark 5.5, this gives us

$$\mathrm{H}^k(\mathbf{CP}^n) = \mathrm{Hom}(\mathrm{H}_k(\mathbf{CP}^n), \mathbf{Z}) = \mathbf{Z}.$$

Thus, the underlying abelian groups in (5.34) are isomorphic. If we pick a generator $\omega \in \mathrm{H}^2(\mathbf{CP}^n)$, we have a ring homomorphism

$$\mathbf{Z}[x] \to \mathrm{H}(\mathbf{CP}^n), x \mapsto \omega,$$

which factors over $\mathbf{Z}[x]/x^{n+1}$, since the $(n+1)$-fold cup product $\omega \cup \cdots \cup \omega \in \mathrm{H}^{2n+2}(\mathbf{CP}^n) = 0$. It suffices to see that $\omega^{\cup n}$ is a generator of $\mathrm{H}^{2n}(\mathbf{CP}^n)(\cong \mathbf{Z})$. If this is the case then also $\omega^{\cup k}$ must be a generator of $\mathrm{H}^{2k}(\mathbf{CP}^n)$.

We will prove this result by a basic, if somewhat special argument due to Lam [Lam70]. We will consider projective space as

$$\mathbf{CP}^n = \{p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, p \neq 0\}/\mathbf{C}^\times,$$

i.e., the space of nonzero complex polynomials of degree $\leqslant n$, up to multiplication by a non-zero complex number.

The proof is based on the map

$$h : M := \mathbf{CP}^1 \times \ldots \times \mathbf{CP}^1 \to \mathbf{CP}^n,$$

(n factors) given by taking products of polynomials as above. Note that $\mathbf{CP}^1 \cong S^2$ is a cell complex with a single cell in dimension 0 and 2. Then M is a cell complex having no cells in odd dimensions, and one cell in dimension 0, and one cell in dimension $2n$ (but more cells in the even intermediate dimensions, e.g. n cells of dimension $2n-2$ etc.) Therefore

$$\mathrm{H}_{2n}(M) = \mathrm{H}^{\mathrm{cell}}_{2n}(M) = \mathbf{Z}.$$

Lemma 5.36. The map

$$h_* : \mathrm{H}_{2n}(M) \to \mathrm{H}_{2n}(\mathbf{CP}^n)$$

maps a generator to $\pm n!$ times a generator. (Both groups are isomorphic to $\mathbf{Z}$.) Therefore, the dual map

$$h^* : \mathrm{H}^{2n}(\mathbf{CP}^n) \to \mathrm{H}^{2n}(M)$$

also maps a generator to $\pm n!$ times a generator.

Proof. Let $D_1, \ldots, D_n$ be pairwise disjoint disks in $\mathbf{CP}^1$ (i.e., parametrizing polynomials of the form $z + \lambda_i \in D_i$, where $\mathbf{C} \supset \{\lambda_i\} \cap \{\lambda_j\} = \varnothing$ for $i \neq j$.) By the fundamental theorem of algebra (Corollary 4.41 + standard abstract algebra), every polynomial of degree n can be factored as a product of linear ones, uniquely up to the order of the factors. Therefore

$$h|_D : D \xrightarrow{\cong} h(D).$$

is a homeomorphism and

$$K := h^{-1}(h(D)) = \bigsqcup_{\sigma \in \Sigma_n} D_\sigma = \bigsqcup_{\sigma \in \Sigma_n} D_{\sigma(1)} \times \ldots \times D_{\sigma(n)}.$$

Here σ ranges over the permutations of n letters.

The inclusions $D_\sigma \subset K$ induce an isomorphism

$$\mathrm{H}_{2n}(M|K) \xrightarrow{\cong} \bigoplus_\sigma \mathrm{H}_{2n}(M|D_\sigma).$$

Indeed, by Proposition 4.50 we can the left hand group as the $2n$-th homology group of the quotient $M/(M\backslash K)$, which space is homeomorphic to a wedge sum of $n!$ copies of S^{2n}. Each of these S^{2n} is homeomorphic to $M/(M\backslash D_\sigma)$.

The map h_* fits into the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}_{2n}(M) \arrow[r, "h_*"] \arrow[d] & \mathrm{H}_{2n}(\mathbf{CP}^n) \arrow[d, "\cong"] \\
\mathrm{H}_{2n}(M|K) \arrow[r, "h_*"] \arrow[d] & \mathrm{H}_{2n}(\mathbf{CP}^n|h(D)) \\
\mathrm{H}_{2n}(M|D_\sigma). \arrow[ur, "h_*", "\cong"'] &
\end{tikzcd}$$

(This diagram is similar to the one in the discussion of the local degree, Lemma 4.44). The right hand vertical map is an isomorphism since $\mathrm{H}_k(\mathbf{CP}^n \backslash h(D)) = 0$ (except for $k = 0$). Indeed, by homotopy invariance we may replace $h(D)$, which is homeomorphic to a product of disks, by a standard disk $B(0,1) \subset \mathbf{C}^n \subset \mathbf{CP}^n$, and then use that the complement $\mathbf{CP}^n \backslash B(0,1)$ is contractible.

By the orientability of M (Lemma 5.28), the compositie $\mathrm{H}_{2n}(M) \to \mathrm{H}_{2n}(M|D_\sigma) \to \mathrm{H}_{2n}(\mathbf{CP}^n|h(D))$ is the *same* map for all $\sigma \in \Sigma_n$. This follows by choosing open balls $B_k \supset D_{\sigma(k)} \cup D_{\sigma'(k)}$, for $k \leqslant n$ and $\sigma, \sigma' \in \Sigma_n$.

A generator of $\mathrm{H}_{2n}(M)$ is mapped to a generator of each $\mathrm{H}_{2n}(M|D_\sigma)$, so that summing up all σ gives the claim. □

Consider the standard inclusion

$$i : \mathbf{CP}^1 \to \mathbf{CP}^n.$$

It induces an isomorphism on H_2 by (4.27). Thus, passing to duals, we see that

$$i^* : \mathrm{H}^2(\mathbf{CP}^n) \to \mathrm{H}^2(\mathbf{CP}^1)$$

maps a generator ω to a generator, which we denote for clarity by $\overline{\omega}$.

In addition, we use the cell structure of M (n cells of dimension 2, no cells of odd dimension) which implies

$$\begin{aligned} \mathrm{H}_2(M) &= \bigoplus_{i=1}^{n} \mathrm{H}_0(\mathbf{CP}^1) \otimes \ldots \otimes \mathrm{H}_2(\mathbf{CP}^1) \otimes \ldots \otimes \mathrm{H}_0(\mathbf{CP}^1) \\ &= \bigoplus_i \mathrm{H}_2(\mathbf{CP}^1). \end{aligned}$$

If we let (for $1 \leqslant k \leqslant n$)

$$\mathbf{CP}^1 \xrightarrow{i_k} M \xrightarrow{p_k} \mathbf{CP}^1$$

be the embedding (i_k) adding the base points in all other factors than the k-th one, respectively (p_k) the projection onto the k-th factor, then we have $p_k \circ i_k = \mathrm{id}$, so that $(p_k)_*(i_k)_* = \mathrm{id}$. Since both groups are isomorphic to $\mathbf{Z}$, this means that $(p_k)_*$ is an inverse to $(i_k)_*$. Passing to duals, we have inverse isomorphisms

$$\mathrm{H}^2(M) \underset{i_k^*}{\overset{p_k^*}{\rightleftarrows}} \bigoplus_{k=1}^{n} \mathrm{H}^2(\mathbf{CP}^1).$$

Lemma 5.37. The map

$$h^* : \mathrm{H}^2(\mathbf{CP}^n) \to \mathrm{H}^2(M)$$

satisfies

$$h^*(\omega) = \sum_k p_k^* \overline{\omega}.$$

Therefore

$$h^*(\omega^{\cup n}) = n! p_1^* \overline{\omega} \cup \cdots \cup p_n^* \overline{\omega}.$$

Proof. In order to show the first claim, it suffices to apply $(i_k)^*$ to both sides and show these agree:

$$(i_k)^* h^* \omega = i^* \omega =: \overline{\omega}.$$

Here we use that the composite

$$\mathbf{CP}^1 \xrightarrow{i_k} M \xrightarrow{h} \mathbf{CP}^n$$

is the standard inclusion i mentioned above. On the other hand,

$$(i_k)^* \sum_r p_r^* \overline{\omega} = \sum_r (p_r^* i_k^*) \overline{\omega} = p_k^* i_k^* \overline{\omega} = \overline{\omega}.$$

Here we use that for $r \neq k$, the map $p_r \circ i_r$ is a constant map, which induces the zero map on $\mathrm{H}_2(\mathbf{CP}^1)$, and therefore also on $\mathrm{H}^2(\mathbf{CP}^1)$.

For the second statement, we use that $f^* : \mathrm{H}(X) \to \mathrm{H}(Y)$ is a *ring homomorphism* for any map $f : Y \to X$. We apply this remark to h, and to the p_k:

$$h^*(\omega^{\cup n}) = (h^*(\omega))^{\cup n}.$$

We have

$$p_k^*(\overline{\omega}) \cup p_k^*(\overline{\omega}) = p_k^*(\underbrace{\overline{\omega} \cup \overline{\omega}}_{\in \mathrm{H}^4(\mathbf{CP}^1)=0}).$$

By the commutativity of the cup product, we also have $p_i^*(\overline{\omega}) \cup p_j^*(\overline{\omega}) = p_j^*(\overline{\omega}) \cup p_i^*(\overline{\omega})$. Using bilinearity of $\cup$ and expanding the sum, we therefore get

$$n! p_1^*(\overline{\omega}) \cup \cdots \cup p_n^*(\overline{\omega}).$$

Thus, if e (resp. f) is a generator of $\mathrm{H}^{2n}(\mathbf{CP}^n)$ (resp. of $\mathrm{H}^{2n}(M)$), and $\omega^{\cup n} = re$ for some $r \in \mathbf{Z}$ then

$$h^*(\omega^n) = h^*(re) = rn!f.$$

On the other hand $\overline{\omega}$ is a generator of $\mathrm{H}^2(\mathbf{CP}^1)$. One can show (this is the so-called Künneth formula) that

$$\mathrm{H}^{2n}(M) = \mathrm{H}^2(\mathbf{CP}^1) \otimes \ldots \otimes \mathrm{H}^2(\mathbf{CP}^1),$$

and that the element $p_1^*(\overline{\omega}) \cup \cdots \cup p_n^*(\overline{\omega}) \in \mathrm{H}^{2n}(M)$ above corresponds to the tensor

$$\overline{\omega} \otimes \ldots \otimes \overline{\omega},$$

which is *also* a generator of the group. Therefore,

$$rn!f = \pm n!(\omega^{\times n}).$$

This implies $r = \pm 1$, finishing the proof of Theorem 5.33.

5.6 The cohomology of $\mathrm{SO}(n)$

In this section, we survey a computation of the cohomology $\mathrm{H}^*(\mathrm{SO}(n), \mathbf{Z}/2)$, following [Hat02, §3.D].

The set

$$\mathrm{O}(n) = O_n(\mathbf{R}) = \{A \in \mathrm{Mat}_{n\times n}(\mathbf{R}) \mid AA^T = \mathrm{id}\}$$

can also be described as the isometries of $\mathbf{R}^n$ fixing the zero vector.

It is a subset of $\mathbf{R}^{n^2}$, and as such a topological group since the multiplication and inverse (in this case given by $A \mapsto A^T$) are continuous. The columns of some $A \in \mathrm{O}(n)$ are vectors in S^{n-1}, which is compact. Therefore

$$\mathrm{O}(n) \subset S^{n-1} \times \ldots \times S^{n-1}$$

is a compact topological group.

Example 5.38. The low-dimensional cases of $\mathrm{SO}(n)$ can be described as follows:

- $\mathrm{SO}(1) = \{\pm 1\}$,
- $\mathrm{SO}(2) = S^1$,

- SO(3) $\cong$ $\mathbf{RP}^3$. To see this, one uses that $\mathbf{RP}^3 = D^3/x \sim -x$ for $x \in \partial D^3$. There is a map $D \to$ SO(3), sending a vector $x \in D^3$ to the rotation by $|x|\pi$ around the line spanned by $x \in \mathbf{R}^3$. For $x \in \partial D$, $|x| = 1$, and so this gives a well-defined map $\mathbf{RP}^3 \to$ SO(3). This map is compact and $\mathbf{RP}^3$ is compact, while SO(3) is Hausdorff. Therefore it is a homeomorphism.

- One can show SO(4) $= S^3 \times$ SO(3).

- SO(8) $= S^7 \times$ SO(7). These latter two are shown using unit vectors of quaternions, resp. octonions.

5.6.1 Basic topological properties

The determinant

$$\det : \mathrm{O}(n) \to \{\pm 1\}$$

is a surjective group homomorphism, therefore its kernel

$$\mathrm{SO}(n) := \ker \det$$

is a subgroup of index 2. We have $\mathrm{SO}(n) = \mathrm{O}(n)\backslash\mathrm{SO}(n)$ (namely if B is an element in the right hand side, then the bijections are given by $A \mapsto AB$, and inverse given by $C \mapsto CB^{-1}$).

- The group SO(n) is path-connected. This can be shown using linear algebra.

- We will later see that SO(n) is a CW complex with a single 0-cell. This gives another argument showing the connectedness of SO(n).

- There is a unique top-dimensional cell, of dimension $\frac{n(n-1)}{2}$.

- SO(n) is orientable, this is a general fact about topological groups.

The computation of (co)homology of SO(n) also gives the one for $\mathrm{O}(n) = \mathrm{SO}(n) \sqcup (\mathrm{O}(n)\backslash\mathrm{SO}(n))$, and also the one for $\mathrm{GL}(n) = \mathrm{O}(n) \times \mathbf{R}^k$, with $k = \frac{n(n+1)}{2} = \dim \mathrm{GL}(n) - \dim \mathrm{O}(n)$. The latter isomorphism is given by Gram–Schmidt orthogonalization, or using polar decomposition.

5.6.2 Cell structure of $\mathrm{SO}(n)$

Recall that a cell complex is inductively obtained by pushouts

$$\begin{tikzcd}
\bigsqcup \partial D^k \arrow[r] \arrow[d] & X_{k-1} \arrow[d] \\
\bigsqcup D^k \arrow[r] & X_k.
\end{tikzcd}$$

The resulting maps $D^k \to X$ are also called the characteristic maps.

For $v \in \mathbf{R}^n \backslash \{0\}$ let $r(v)$ be the reflection along $\langle v \rangle^\perp \in \mathrm{O}(n) \backslash \mathrm{SO}(n)$. Then we set

$$\rho(v) = r(v) r(e_1) \in \mathrm{SO}(n).$$

This defines a map $\rho : \mathbf{R}^n \to \mathrm{SO}(n)$. However, $\rho(v)$ only depends on the line spanned by v, so we have a map

$$\rho : \mathbf{P}^{n-1} \to \mathrm{SO}(n).$$

This map is continuous and injective. This can be restricted to give subspaces $\mathbf{P}^j \subset \mathrm{SO}(n)$, for $j \leqslant n-1$.

For a multi-index $I = (i_1, \dots, i_m)$ (with all $i_j \leqslant n-1$), we have a map

$$\rho : \mathbf{P}^I := \mathbf{P}^{i_1} \times \dots \times \mathbf{P}^{i_m} \to \mathrm{SO}(n), (v_1, \dots, v_m) \mapsto \rho(v_1) \cdot \dots \cdot \rho(v_m).$$

Definition 5.39. We call such a multi-index admissible if $I = (0)$ or if $n > i_1 > i_2 > \dots > i_m > 0$.

Recall that $\mathbf{P}^k$ has a cell structure with one cell in each dimension in $[0, k]$. In particular, there is a map $\phi^k : D^k \to \mathbf{P}^k$. Their product is a map

$$\phi^I : D^I := \prod_j D^{i_j} \to \mathbf{P}^I := \prod \mathbf{P}^{i_j}.$$

The composition with ρ gives a map

$$\rho\phi^I : D^I \to \mathrm{SO}(n).$$

Proposition 5.40. The maps $\rho\phi^I$, for all the admissible multi-indices I, are the characteristic maps of a cell structure on $\mathrm{SO}(n)$.

Remark 5.41. The admissible I will contribute cells of dimension $\sum_j i_j$. In particular, there is a single 0-cell. Also, the multi-index

$I = (n_1, \dots, 1)$ gives a single top-dimensional cell, of dimension $\frac{n(n-1)}{2}$.

In total, there are $2^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i}$ admissible sequences.

Example 5.42. For $n = 3$, there are the following multi-indices and associated characteristic maps:

- $(0) \rightsquigarrow D^0 \to \mathrm{SO}(3)$.
- $(1) \rightsquigarrow D^1 \to \mathrm{SO}(3)$.
- $(2) \rightsquigarrow D^2 \to \mathrm{SO}(3)$.
- $(2, 1) \rightsquigarrow D^2 \times D^1 (\cong D^3) \to \mathrm{SO}(3)$.

These cells match the standard cell structure on $\mathbf{RP}^3$.

Proof. (Sketch:) The proof uses a general criterion for cell structures. One needs to check the following (for all admissible I):

(1) $\rho\phi^I$ induces a homeomorphism $(D^I)^\circ$ onto its image. Let e^I be the image thereof.

(2) All e^I are disjoint and cover $\mathrm{SO}(n)$.

(3) $\rho\phi^I(\partial D^I)$ is contained in a union of lower-dimensional cells.

If we let $p : \mathrm{SO}(n) \to S^{n-1}$, $\alpha \mapsto \alpha(e_n)$. Then

$$\mathbf{P}^{n-1} \backslash \mathbf{P}^{n-2} \underset{\cong}{\xrightarrow{p}} S^{n-1} \backslash \{e_n\}.$$

We have a homeomorphism

$$h : (\mathbf{P}^{n-1} \backslash \mathbf{P}^{n-2}) \times \mathrm{SO}(n-1) \xrightarrow{\cong} \mathrm{SO}(n) \backslash \mathrm{SO}(n-1),$$

given by $(v, \alpha) \mapsto \rho(v)\alpha$, with inverse $\beta \mapsto (v_\beta, \alpha_\beta)$, with v_β corresponding to $p(\beta)$ under the above homeomorphism, and $\alpha_\beta = \rho(v_\eta)^{-1}\beta(e_n)$.

Thus, for $I = (n-1, \dots)$, we get a cell (induced by $(D^{n-1})^\circ \times ((D^{i_2})^\circ \times \dots \times (D^{i_m})^\circ)$ inside $\mathrm{SO}(n) \backslash \mathrm{SO}(n-1)$.

For the third point above, note that

$$\partial D^I = \partial D^{i_1} \times D^{i_2} \times \dots \times D^{i_m} + D^{i_1} \times \partial D^{i_2} \times \dots + \dots .$$

For $I = (i_1, \dots, i_m)$, the multi-index $(i_1 - 1, i_2, \dots, i_m)$ need not be admissible. But, one can show (where the juxtaposition denotes the product of subsets of $\mathrm{SO}(n)$)

$$\mathbf{P}^i \mathbf{P}^i \subset \mathbf{P}^i \mathbf{P}^{i-1}.$$

Therefore, one can replace the possibly non-admissible multi-index $(i_1 - 1, i_2, \dots)$ by $(i_1 - 1, i_2 - 1, \dots)$. $\square$

Corollary 5.43. The map

$$\rho : \mathbf{P}^{n-1} \times \ldots \times \mathbf{P}^1 \to \mathrm{SO}(n)$$

is surjective and cellular.

5.6.3 $\mathbf{Z}/2$-homology and cohomology

According to computations above, the cellular complex for $\mathbf{P}^I$ are complexes with differential equal to 0, and in each degree k given by $(\mathbf{Z}/2)^{a_k}$, with k being the number of k-dimensional cells in $\mathbf{P}^I$. The surjectivity of ρ implies that likewise the differential in the cellular complex for $\mathrm{SO}(n)$ is is zero.

We want to use an isomorphism

$$\mathrm{H}_k(\mathrm{SO}(n), \mathbf{Z}/2) = \mathrm{H}_k(S^{n-1} \times \ldots \times S^1, \mathbf{Z}/2).$$

For this, we need to understand the homology of the right hand side. We do this using the following

Theorem 5.44. If X, Y are cell complexes, R is principal ideal domain, then there is a short exact sequence

$$0 \to \bigoplus_i \mathrm{H}_i(X, R) \otimes_R \mathrm{H}_{n-i}(Y, R) \xrightarrow{h} \mathrm{H}_n(X \times Y, R) \to \bigoplus_i \mathrm{Tor}_R(\mathrm{H}_i(X, R), \mathrm{H}_{n-i-1}(Y, R) \to 0.$$

In particular, if R is a field, such as $\mathbf{Z}/2$, then h is an isomorphism.

Here is the description of the cup product:

Theorem 5.45.

$$\mathrm{H}^*(\mathrm{SO}(n), \mathbf{Z}/2) = \bigoplus_i \mathrm{H}^i(\mathrm{SO}(n), \mathbf{Z}/2) = \bigotimes_{i \text{ odd}} \mathbf{Z}/2[\beta_i]/\beta_i^{p_i}.$$

Here $|\beta_i| = i$, i.e., $\beta_i \in \mathrm{H}^i$. This is the duall class of e_i, which is the i-dimensional cell of $\mathbf{P}^{n-1} \subset \mathrm{SO}(n)$. Moreover, $p_i = \min_k\{2^k$ such that $2^k i \geqslant n\}$.

Example 5.46. For $n = 3$, $|\beta_1| = 1$, so that $p_1 = 4$. $|\beta_3| = 3$, then $p_3 = 1$, so that $\mathbf{Z}/2[\beta_3]/(\beta_3) = \mathbf{Z}/2$. Thus

$$\mathrm{H}^*(\mathrm{SO}(3), \mathbf{Z}/2) = \mathbf{Z}_2[\beta_1]/\beta_1^4.$$

This agrees with $\mathrm{H}^*(\mathbf{RP}^3, \mathbf{Z}/2) = \mathbf{Z}/2[\alpha]/\alpha^4$, as seen above.

5.7 Exercises

Exercise 5.1. Let X be an orientable compact manifold of *odd* dimension. Show that its *Euler characteristic* vanishes:

$$\chi(X) = 0.$$

Appendix A

Category theory

A.1 The Yoneda lemma

Given any category C, and any object $X \in C$, there are functors

$$h_X : C^{\mathrm{op}} \to \mathrm{Set}, Y \mapsto \mathrm{Hom}_C(Y, X).$$

The functor h_X is called the *representing functor* associated to X.

Let C be any small category, $X \in C$ and $F : C^{\mathrm{op}} \to \mathrm{Set}$ a functor. There is a map (of sets)

$$F(X) \to \mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{Set})}(h_X, F)$$

that sends $f \in F(X)$ to the natural transformation $h_X \to F$ that is given on objects $Y \in C^{\mathrm{op}}$ by $h_X(Y) = \mathrm{Hom}_C(Y, X) \ni \alpha \mapsto F(\alpha)(f) \in F(Y)$. (Note here $F(\alpha) : F(X) \to F(Y)$ since F is contravariant. Note also that given a morphism $Y \to Z$ in C^{op}, i.e., a morphism $y : Z \to Y$ in C, the diagram

$$\begin{tikzcd}
h_X(Y) = \mathrm{Hom}_C(Y, X) \arrow[r] \arrow[d, "y^* = \mathrm{Hom}_C(y, X)"] & F(Y) \arrow[d, "F(y)"'] \\
h_X(Z) = \mathrm{Hom}_C(Z, X) \arrow[r] & F(Z)
\end{tikzcd}$$

commutes since F is a functor. Thus we have indeed defined an element in $\mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{Set})}(h_X, F)$.)

Conversely, there is a map (of sets)

$$\mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Set})}(h_X, F) \to F(X)$$

that sends a natural transformation $g : h_X \to F$ to $g(\mathrm{id}_X) \in F(X)$ (note the evaluation of g at X gives a map $h_X(X) = \mathrm{Hom}_C(X, X) \ni \mathrm{id}_X \to g(\mathrm{id}_X) \in F(X)$).

Lemma A.1. The above two maps are inverse to each other, so we have an isomorphism.

The (completely formal) proof is left as an exercise.

We now specialize this assertion. There is a natural map

$$\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Set})}(h_X, h_Y). \tag{A.2}$$

It sends a morphism $f : X \to Y$ to the morphism $h_X \to h_Y$ whose evaluation on any $T \in C$ is given by

$$h_X(T) = \mathrm{Hom}_C(T, X) \to h_Y(T) = \mathrm{Hom}_C(T, Y), t \mapsto f \circ t$$

Lemma A.3. The above map (A.2) is a bijection.

In more high-level language: for any small category C, the functor

$$C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}), X \mapsto h_X := \mathrm{Hom}_C(-, X)$$

is fully faithful. It is called the *Yoneda embedding*.

Proof. This follows directly from Lemma A.1:

$$\mathrm{Hom}_C(X, Y) = h_Y(X) = \mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}},\mathrm{Set})}(h_X, h_Y).$$

Appendix B

Homological algebra

B.1 The tensor product

The tensor product

$$A \otimes B := A \otimes_{\mathbf{Z}} B$$

of two abelian groups is characterized by the following universal property: it is an abelian group together with a group homomorphism

$$A \times B \to A \otimes B,$$

such that every *bilinear* map $f : A \times B \to C$, where C is an arbitrary abelian group, factors uniquely like so:

$$\begin{tikzcd} A \times B \arrow[r] \arrow[dr, "f"'] & A \otimes B \arrow[d, dotted] \\ & C. \end{tikzcd}$$

We will mostly apply the tensor product in the case where A and B are free, i.e., there are isomorphisms

$$A \cong \bigoplus_{i \in I} \mathbf{Z}, B \cong \bigoplus_{j \in J} \mathbf{Z}$$

for appropriate (possibly infinite) sets I, J. In this case,

$$\bigoplus_{i \in I} \mathbf{Z} \otimes \bigoplus_{j \in J} \mathbf{Z} \cong \bigoplus_{(i,j) \in I \times J} \mathbf{Z}. \tag{B.1}$$

More generally, let R be a commutative ring. Then there is the tensor product

$$M \otimes_R N,$$

which is again an R-module, and which satisfies the same universal property as above:

$$\mathrm{Hom}_R(M \otimes_R N, K) = \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, K)),$$

where Hom_R denotes the R-module consisting of R-linear maps.

B.1.1 Flatness

For a fixed R-module M, the tensor product functor $N \mapsto M \otimes_R N$ is a right-exact functor, i.e., for any short exact sequence of the form

$$N_1 \to N_2 \to N_3 \to 0,$$

the sequence

$$M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

is again exact. (Indeed, $- \otimes_R N$ is left adjoint to $\mathrm{Hom}_R(N, -)$, so the former preserves all colimits including the above one: $N_3 = \mathrm{coker}(N_1 \to N_2)$. See, e.g., [Eis95, Appendix 5].)

M is called a *flat* R-module, if the functor $M \otimes_R -$ is exact (equivalently: if $N_1 \subset N_2$ is a submodule, then $M \otimes N_1 \subset M \otimes N_2$ is again a submodule). Every free R-module, and more generally every projective R-module is flat. In general, these implications are not reversible, but for $R = \mathbf{Z}$ the situation does simplify: an abelian group is free abelian iff it is a projective $\mathbf{Z}$-module iff it is a flat $\mathbf{Z}$-module.

A non-flat $\mathbf{Z}$-module is $\mathbf{Z}/n$ for $n > 0$: the injective map $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$ becomes, after tensoring with $\mathbf{Z}/n$, the map $\mathbf{Z}/n \xrightarrow{n} \mathbf{Z}/n$, but in $\mathbf{Z}/n$ multiplication by n is not injective.

Any localization of a flat module is again flat. For example $\mathbf{Q} = \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \dots]$ is a flat module (but is not projective).

For any ring R, a flat-module M is torsion-free (i.e., multiplication by $r \neq 0$ is injective on M). If R is a principal ideal domain, then the converse holds as well.

B.1.2 Tor functors

Tor functors (the name comes from torsion, which is motivated by the above example) provide a way to measure how "non-flat" modules are. For a proof of the following statement, see e.g., [Eis95, §6.2].

Proposition B.2. There are functors, for any $i \geqslant 0$,

$$\mathrm{Tor}_i^R : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R$$

such that
(1) $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$.
(2) If

$$\dots P_1 \to P_0 \to M \to 0 \tag{B.3}$$

is a *projective resolution* (i.e., an exact complex with P_k being projective modules) or, more generally a *flat resolution* (the P_k are flat R-modules), then

$$\mathrm{Tor}_i^R(M, N) = \mathrm{H}_i(P_* \otimes_R N).$$

(3) In particular, if M is projective or flat, then $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$ and $N \in \mathrm{Mod}_R$.
(4) For a short exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

there is a long exact sequence

$$\dots \to \mathrm{Tor}_2^R(M, N_3) \to \mathrm{Tor}_1^R(M, N_1) \to \mathrm{Tor}_1^R(M, N_2) \to \mathrm{Tor}_1^R(M, N_3) \to M \otimes_R N_3 \to M \otimes_R N_2 \to$$

The Tor functors also satisfy a symmetry

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^R(N, M),$$

extending the isomorphisms $M \otimes_R N \cong N \otimes_R M$.

One also refers to the collection of all the Tor functors as the *left derived functor* of $\otimes$.

For a principal ideal domain R (e.g., $R = \mathbf{Z}$), every submodule of a free R-module is free, so that N admits a free resolution of the form

$$0 \to P_1 \to P_0 \to M \to 0. \tag{B.4}$$

This implies that all Tor_i^R vanish for $i \geqslant 2$.

Proposition B.5. (*Künneth formula*) Let R be a principal ideal domain (such as $R = \mathbf{Z}$), $C, D \in \mathrm{Ch}(\mathrm{Mod}_R)$ be two complexes, with each $C_n \in \mathrm{Mod}_R$ being flat. Then there is a short exact sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(C) \otimes \mathrm{H}_q(D) \to \mathrm{H}_n(C \otimes D) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(\mathrm{H}_p(C), \mathrm{H}_q(D)) \to 0. \tag{B.6}$$

Example B.7. For $D = \mathbf{Z}/\ell$ (concentrated in degree 0), this gives back the sequence

$$0 \to \mathrm{H}_n(C)/\ell \to \mathrm{H}_n(C/\ell) \to (\mathrm{H}_{n-1}(C))_\ell \to 0,$$

of Example 4.14 (cf. also the remarks after Definition 3.16): indeed,

$$\mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/\ell, M) = M_\ell,$$

by virtue of the resolution $0 \to \mathbf{Z} \xrightarrow{\ell} \mathbf{Z} \overset{\mathbf{Z}}{/} \ell \to 0$, which gives after tensoring with M:

$$0 = \mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/\ell, \mathbf{Z}) \to \mathrm{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/\ell, M) \to M \otimes_{\mathbf{Z}} \mathbf{Z} \xrightarrow{\ell} M \otimes_{\mathbf{Z}} \mathbf{Z} \to M \otimes_{\mathbf{Z}} \mathbf{Z}/\ell \to 0.$$

By comparison $D = \mathbf{Q}$ (again in degree 0) is flat, so $\mathrm{Tor}_1^{\mathbf{Z}}(-, \mathbf{Q}) = 0$, and we get isomorphisms

$$\mathrm{H}_n(C) \otimes \mathbf{Q} \xrightarrow{\cong} \mathrm{H}_n(C \otimes \mathbf{Q}).$$

Proof. We first do the special case where the differentials in C are zero. In this case $C = \bigoplus_{i \in \mathbf{Z}} C_i[i]$ and $\mathrm{H}_p(C) = C_p$ is a free R-module and therefore $\mathrm{Tor}_1^R(\mathrm{H}_p(C), \mathrm{H}_q(D)) = 0$. Our claim now holds since H_* commutes with tensoring with a *flat* module, and also commutes with direct sums:

$$\begin{aligned}
\bigoplus_{p+q=n} C_p \otimes \mathrm{H}_q(D) &= \bigoplus \mathrm{H}_q(C_p \otimes D) \\
&= \bigoplus_{p+q=n} \mathrm{H}_{p+q}(C_p[p] \otimes D) \\
&= \mathrm{H}_n(\bigoplus C_p[p] \otimes D) \\
&= \mathrm{H}_n(C \otimes D).
\end{aligned}$$

We now do the general case. Let $B_i \subset Z_i \subset C_i$ be the boundaries and cycle submodules in C_i. We use that R is a PID, so that "flat" is equivalent to "torsion-free". In particular, B_i and Z_i are also flat. These groups form subcomplexes $B \subset Z \subset C$ with the property that their differentials are zero, and there is a short exact sequence

$$0 \to Z \to C \xrightarrow{d} B[1] \to 0.$$

Tensoring with D gives an exact sequence, since B_i are flat (so that $\mathrm{Tor}_1^R(B_i, D_n) = 0$):

$$0 \to Z \otimes D \to C \otimes D \xrightarrow{d} B[1] \otimes D \to 0.$$

We have long exact sequences of homology groups (the $\oplus$ run over $p+q=n$), where the vertical isomorphisms come from the special case above:

$$\begin{tikzcd}
0 \arrow[r] & \oplus\mathrm{Tor}_1(\mathrm{H}_pC, \mathrm{H}_qD) \arrow[r] & \oplus B_p \otimes \mathrm{H}_q(D) \arrow[r] \arrow[d, "\cong"] & \oplus Z_p \otimes \mathrm{H}_q(D) \arrow[r] \arrow[d, "\cong"] & \oplus \mathrm{H}_pC \otimes \mathrm{H}_qD \arrow[r] \arrow[d, dotted] & 0 \\
 & \dots \arrow[r] & \mathrm{H}_{n+1}(B \otimes D) \arrow[r] & \mathrm{H}_n(Z \otimes D) \arrow[r, "i"] & \mathrm{H}_n(C \otimes D) \arrow[r, "\partial_C \otimes \mathrm{id}_D"] & \mathrm{H}_{n+1}(B \otimes
\end{tikzcd}$$

There is a unique dotted map as indicated. It is injective by a diagram-chase. The dotted map has the same image as the one labelled i, so that their cokernels agree as well. By the 0 at the very top left (which holds because of $0 = \mathrm{Tor}_1(Z_p, \mathrm{H}_qD)$, using that Z_p is flat), we have $\operatorname{coker} i = \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(\mathrm{H}_pC, \mathrm{H}_qD)$. $\square$

Remark B.8. If the C_n are in fact projective modules, then one can choose non-canonical isomorphisms $C_n \cong Z_n \oplus B_n$, which gives the added information that (B.6) is in fact a *split* exact sequence, i.e., there are isomorphisms

$$\bigoplus_{p+q=n} \left(\mathrm{H}_p(C) \otimes \mathrm{H}_q(D) \oplus \mathrm{Tor}_1^R(\mathrm{H}_{p-1}C, \mathrm{H}_qD)\right) \cong \mathrm{H}_n(C \otimes D).$$

However, this depends on the above choices and is therefore not functorial in C. See, for example, [Rot88, Corollary 10.82].

B.2 Exercises

Exercise B.1. Let

$$\dots \to 0 \to A \stackrel{a}{\to} B \stackrel{b}{\to} C \to 0 \dots$$

be an exact complex of abelian groups. (This is called a *short exact sequence*.) Show that for any abelian group T, there are complexes, with appropriate natural maps

$$0 \to \mathrm{Hom}_{\mathrm{Ab}}(T, A) \stackrel{a_*}{\to} \mathrm{Hom}_{\mathrm{Ab}}(T, B) \stackrel{b_*}{\to} \mathrm{Hom}_{\mathrm{Ab}}(T, C) \to 0.$$

Show that this complex is exact except possibly at the spot $\mathrm{Hom}(T, C)$, i.e., b_* need not be surjective. Show that for a *free* abelian group T ($T = \mathbf{Z}[S]$ for some set S), the complex is exact.

Also show that

$$0 \to A \otimes T \to B \otimes T \to C \otimes T \to 0$$

is a complex. Show that it is exact except that possibly the map $A \otimes T \to B \otimes T$ need not be injective. Show that the complex is exact for a free abelian group T.

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