Algebraic Geometry

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These are notes for a course named "Algebraic Geometry 2" given at the Università degli studi di Padova in 2025.

The lecture notes have been prepared using various sources including: [Stacks], [Vak17], [GW20], [Har83], [Eis95], [Mat80], [Gro61], [Sch16], [KN21].

Contents

Chapter 1

Rings and their spectra

All theorems in algebraic geometry are ultimately grounded in commutative algebra, or the theory of commutative rings (and their modules). In this chapter, we study a few ring-theoretic notions, and introduce the Zariski topology on the spectrum of a ring.

Convention 1.0.1. Throughout, all rings are commutative, associative and unital. We use A to denote a ring and k for a field.

The following definition opens the door from commutative algebra to algebraic geometry.

Definition 1.0.2. The *spectrum* of A is the set

$$\operatorname{Spec} A := \{\mathfrak{p} \subset A \text{ prime ideal}\}.$$

For a map $f : A \to B$ of rings, we have an induced map (of sets, for now),

$$\operatorname{Spec} B \to \operatorname{Spec} A, \mathfrak{q}(\subset B) \mapsto f^{-1}(\mathfrak{q}), \tag{1.0.3}$$

noting that $f^{-1}(\mathfrak{q})$ is a prime ideal in A. We will denote this map by $\operatorname{Spec} f$ or just f. Sometimes we also use a different letter such as φ in order to avoid an overload of notation.

1.1 The Zariski topology

For $f \in A$, we denote

$$D(f) := \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}\}.$$

Recall that these prime ideals are precisely the prime ideals of the localization $A[f^{-1}]$ (more precisely, the map $\operatorname{Spec}(A[f^{-1}]) \to \operatorname{Spec} A$ from (1.0.3) is injective and its image is $D(f)$.)

Definition 1.1.1. The *Zariski topology* on $\operatorname{Spec} A$ is defined to be the topology generated by the subsets $D(f)$. These subsets are called *basic open subsets*.

Remark 1.1.2. Since $D(f) \cap D(g) = D(fg)$, the open subsets in $\operatorname{Spec} A$ are therefore (possibly infinite) unions of open subsets of the basic open subsets. We have $D(0) = \varnothing$ and $D(1) = \operatorname{Spec} A$.

Lemma 1.1.3. If $f : A \to B$ is a ring map, then $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous (for the Zariski topology). More precisely $\varphi^{-1}(D(a))$ (for some $a \in A$) equals the fundamental open subset $D(f(a)) \subset \operatorname{Spec} B$.

Proof. This is directly clear from the definitions. $\square$

The closed subsets of this topology are the of the form

$$V(M) := \{\mathfrak{p} \subset A \mid M \subset \mathfrak{p}\},$$

where $M \subset A$ is an arbitrary subset. Indeed, $V(M) = \bigcap_{f \in M} V(\{f\})$, and $V(\{f\})$ is the complement of $D(f)$. In the above it is enough to consider M to be an ideal. Indeed, if $I \subset A$ denotes the ideal generated by M, then

$$V(M) = V(I).$$

We can make a more precise relation between closed subsets of $\operatorname{Spec} A$ and (certain) ideals of A as follows. Recall that the *radical* $\sqrt{I}$ of an ideal I is defined as

$$\sqrt{I} := \{a \in A \mid a^n \in I \text{ for } n \gg 0\}. \tag{1.1.4}$$

A basic statement of commutative algebra [Stacks, Tag 00E0] asserts

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I, \mathfrak{p} \text{ a prime ideal}} \mathfrak{p}. \tag{1.1.5}$$

As a special case, the *nil-radical* is

$$\sqrt{0} := \{a \in A \mid a^n = 0 \text{ for } n \gg 0\} = \bigcap_{\mathfrak{p} \text{ prime ideal}} \mathfrak{p}.$$

For a subset $Y \subset \operatorname{Spec} A$, we define

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

This is clearly an ideal; one should think of this as the ideal of those elements $f \in A$ "vanishing" at all points in Y in the sense explained in (1.2.5) below.

Lemma 1.1.6. There are mutually inverse bijections

$$\left\{J \subset A \text{ ideal } \mid J = \sqrt{J}\right\} \underset{I(-)}{\overset{V(-)}{\rightleftarrows}} \{Y \subset \operatorname{Spec} A \text{ closed}\}.$$

Slightly more generally and precisely:
(1) $I(Y) = \sqrt{I(Y)}$ for any (not necessarily closed) subset $Y \subset \operatorname{Spec} A$.

(2) $I(V(J)) = \sqrt{J}$ for any ideal J.

(3) $V(I(Y)) = \overline{Y}$ (the closure).

Proof. (1): We have $\mathfrak{p} = \sqrt{\mathfrak{p}}$ for any prime ideal, and also for any intersection of prime ideals, such as $I(Y)$. (2): This is a reformulation of (1.1.5). (3): A closed subset $V(K)$ (for some ideal K) contains Y iff K is contained in the prime ideals $\mathfrak{p}$ belonging to Y, which happens iff $K \subset I(Y)$. The stated bijection is then a consequence of (2) and (3). □

The Zariski topology is very different from topological spaces such as $\mathbf{R}^n$, as we will soon understand. In the sequel, we will define several basic properties of a topological space and then rephrase them in ring-theoretic terms.

Definition 1.1.7. Let X be a topological space, and $x, y \in X$.

- We call X *quasi-compact* if for every open covering $X = \bigcup_{i \in I} U_i$ there is a finite subcovering, i.e., already finitely many of the U_i cover X.
- We say x is a *generic point* if $\overline{\{x\}} = X$. This is equivalent to requiring x to be contained in any non-empty open subset $U \subset X$.

- We say x is a *closed point* if $\overline{\{x\}} = \{x\}$. (Note this is the other extreme in comparison to a generic point.)

- We write $x \rightsquigarrow y$ if $y \in \overline{\{x\}}$, i.e., y lies in the closure of x. We say that y is a *specialization* of x (or x is a *generalization* of y) in this case.

Lemma 1.1.8. (1) We have $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ for any prime ideal $\mathfrak{p}$.

(2) $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ iff $\mathfrak{p} \subset \mathfrak{q}$. In particular:

- $\mathfrak{p}$ is a generic point in $\operatorname{Spec} A$ iff $\mathfrak{p} = \sqrt{0}$ (the nilradical).
- $\mathfrak{p}$ is a closed point iff $\mathfrak{p}$ is a maximal ideal.

Proof. (1) holds directly by definition. This implies the other claims as well: $\mathfrak{p}$ is a generic point iff $\mathfrak{p}$ is contained in *every* prime ideal, i.e., $\mathfrak{p} \subset \bigcap_{\mathfrak{q} \text{ prime}} \mathfrak{q} = \sqrt{0}$. However, for any prime ideal, we always have $\sqrt{0} \subset \mathfrak{p}$, so the preceding containment is actually equivalent to $\mathfrak{p} = \sqrt{0}$. □

Remark 1.1.9. Unlike many topological spaces encountered in other branches of mathematics, $\operatorname{Spec} A$ is very rarely Hausdorff (and therefore compact in the sense of usual point-set topology); we will describe this precisely in Lemma 1.4.9.

Lemma 1.1.10. (1) For a subset $S \subset A$, $V(S) = \varnothing$ iff S generates the unit ideal, i.e., if $1 = \sum s_i t_i$ for appropriate $s_i \in S$ and $t_i \in A$.

(2) Suppose a set of elements $f_i \in A$ is fixed. Then $\bigcup_i D(f_i) = \operatorname{Spec} A$ iff the f_i generate the unit ideal.

(3) For $f, g \in A$ we have $D(f) \subset D(g)$ iff g is a unit in $A[f^{-1}]$ (in which case there is a ring homomorphism $A[g^{-1}] \to A[f^{-1}]$). In particular,

$$D(f) = D(g) \Leftrightarrow A[f^{-1}] = A[g^{-1}]. \tag{1.1.11}$$

(4) $\operatorname{Spec} A$ is quasi-compact.

Proof. For (1) we note that $V(S) = V(I)$ where I is the ideal generated by S. An ideal I is the unit ideal precisely if it is not contained in any maximal ideal.

(2) follows from (1) by passing to the open complements.

(3): We have $D(f) \subset D(g)$ iff $D(f) = D(f) \cap D(g)$ i.e., iff $D(f) = D(fg)$. We note that $D(f) = \operatorname{Spec} A[f^{-1}]$. Thus, applying (2) to the ring $A[f^{-1}]$, the preceding condition holds iff fg generates the unit ideal in $A[f^{-1}]$ which happens iff g is a unit in $A[f^{-1}]$.

For (4), it is enough to show that any covering $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$ by basic open subsets admits a finite subcovering. This condition is equivalent to $\bigcap_i V(\{f_i\}) = \varnothing$. This implies the claim by (1). Indeed, any element in the ideal generated by the f_i is a *finite* linear combination of the f_i. □

Lemma 1.1.12. Let $f : A \to B$ be a ring homomorphism and $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ the induced map. Let $J \subset B$ be an ideal. Then

$$V(f^{-1}(J)) = \overline{\varphi(V(J))}. \tag{1.1.13}$$

In particular φ is *dominant* (i.e., its image $\operatorname{im} \varphi$ is *dense*) iff every element of $\ker f$ is nilpotent.

In particular, if A is reduced then f is injective iff φ has dense image.

Proof. We have the following equalities of ideals in A:

$$I(\varphi(V(J))) := \bigcap_{\mathfrak{p} \in \varphi(V(J))} \mathfrak{p} = \bigcap_{\mathfrak{q} \in V(J)} f^{-1}(\mathfrak{q}) = f^{-1}\left(\bigcap_{\mathfrak{q} \in V(J)} \mathfrak{q}\right) \overset{(1.1.5)}{=} f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}.$$

We then apply $V(-)$ and conclude the assertion in (1.1.13) using Lemma 1.1.6(3) for the right hand side.

For the next claim, take $J = 0$, and note that for an ideal $I \subset A$ the inclusion $\operatorname{Spec} A/I \subset \operatorname{Spec} A$ is an equality precisely if I is contained in the nilradical (Exercise 1.1.22(4)). □

Definition 1.1.14. A topological space is called *Noetherian* if the descending chain condition holds for closed subsets, i.e., if any sequence

$$X \supset V_1 \supset V_2 \supset \dots$$

satisfies $V_n = V_{n+1} = \dots$ for large enough n.

Lemma 1.1.15. If A is a Noetherian ring, then $\operatorname{Spec} A$ is a Noetherian topological space in the sense above.

Definition 1.1.16. A topological space X is called *irreducible* if $X \neq \varnothing$ and whenever

$$X = V \cup W$$

for two *closed* subsets V and W, necessarily there holds $X = V$ or $X = W$.

A subset $Z \subset X$ is called an *irreducible component* if it is a maximal irreducible subset of X.

Lemma 1.1.17. (1) The irreducible closed subsets of $\operatorname{Spec} A$ are exactly the subsets $V(\mathfrak{p}) (= \overline{\{\mathfrak{p}\}})$, for arbitrary prime ideals $\mathfrak{p}$.

(2) The irreducible components of $\operatorname{Spec} A$ are exactly the subsets $V(\mathfrak{p})$ with $\mathfrak{p}$ being a *minimal* prime ideal.

In particular, if A is a domain (i.e., has no zero-divisors), so that (0) is a prime ideal, then $\operatorname{Spec} A$ is irreducible.

Exercises

Exercise 1.1.18. (1) If X is a Noetherian topological space and $U \subset X$ open, prove that U is quasi-compact.

(2) Let A be a Noetherian ring. Show that any open subset $U \subset \operatorname{Spec} A$ is necessarily a *finite* union of basic open subsets, i.e., $U = \bigcup_{i=1}^n D(f_i)$.

(3) Let $A = \mathbf{Z}[t_1, t_2, \dots]$ (countably many variables). Show that $U := \operatorname{Spec} A \backslash V((t_1, t_2, \dots))$ can not be covered by finitely many basic open subsets. In the parlance of Definition 1.6.20 below, one may think of $\operatorname{Spec} A$ as an *infinite-dimensional affine space*, and U the complement of the origin in there.

Exercise 1.1.19. Let $I \subset A$ be an ideal. Prove that $V(I) = V(\sqrt{I})$, where $\sqrt{I} := \{a \in A, a^n \in I\}$ denotes the *radical* of I.

Exercise 1.1.20. Let X be a topological space.

(1) If X is irreducible and $U \subset X$ open, prove that U is irreducible.

(2) Prove that X is irreducible iff for any open $\varnothing \neq U, V \subset X$ one has $U \cap V \neq \varnothing$ (i.e., any two open subsets intersect, unless one of them is empty).

Exercise 1.1.21. (Solution at p. 105)

(1) Prove that a subset $Y \subset \operatorname{Spec} A$ is irreducible iff $I(Y)$ is a prime ideal.

(2) Deduce that $\operatorname{Spec} A$ is irreducible iff the *nilradical* $\sqrt{0}$ is a prime ideal in A.

(3) One of the following three is *reducible* (i.e., not irreducible). Which one? $\operatorname{Spec} \mathbf{Z}[x,y]/x^2 - y^2$, $\operatorname{Spec} \mathbf{Z}[x,y]/x^2 - y^3$, $\operatorname{Spec} \mathbf{Z}[x,y]/xy - 1$. (In particular, this shows that a closed subscheme of $\operatorname{Spec} \mathbf{Z}[x,y]$, which is irreducible, may be reducible. This is in contrast with the permanence of irreducibility for open subsets proved in Exercise 1.1.20.)

Exercise 1.1.22. Let A be a ring, $f \in A$ and $I \subset A$ an ideal.

(1) Prove that the map $\operatorname{Spec} A[f^{-1}] \to D(f)$ is a homeomorphism, where we endow $D(f)$ with the subspace topology of $\operatorname{Spec} A$. Slightly more generally, prove that for a multiplicatively closed subset $S \subset A$, there is a homeomorphism (where the right hand side carries the subspace topology of $\operatorname{Spec} A$)
$$\operatorname{Spec} A[S^{-1}] \to \{\mathfrak{p} \in \operatorname{Spec} A, \mathfrak{p} \cap S = \varnothing\}.$$

(2) Prove that the map $\operatorname{Spec} A/I \to V(I)$ is a homeomorphism, where we endow $V(I)$ with the subspace topology of $\operatorname{Spec} A$.

(3) Deduce that there is a homeomorphism
$$\operatorname{Spec}(A_{\mathrm{red}}) := \operatorname{Spec}(A/\sqrt{\{0\}}) \to \operatorname{Spec} A, \tag{1.1.23}$$
where $\sqrt{\{0\}}$ denotes the *nil-radical* of A.

(4) Strengthen the previous assertion as follows: for an ideal $I \subset A$ the inclusion $\operatorname{Spec} A/I \subset \operatorname{Spec} A$ is an equality precisely if $I \subset \sqrt{0}$.

Exercise 1.1.24. Show that the converse of Lemma 1.1.15 fails (e.g., using Exercise 1.1.22).

Exercise 1.1.25. (Solution at p. 105) Recall that the *boundary* of an open subset $U \subset X$ in some topological space is defined as
$$\partial U := \overline{U} \backslash U.$$
Let $X = \operatorname{Spec} A$ and $U = D(f)$ for some $f \in A$. Establish a bijection
$$\partial U = \operatorname{Spec} A/I,$$
where I is the ideal generated by f and $\sqrt{0} : (f) := \{r \in A | rf \in \sqrt{0}\} = \{r \in A | (rf)^n = 0 \text{ for some } n \gg 0\}$.

Hint: prove that $D(r) = \varnothing$ iff $r \in \sqrt{0}$.

Exercise 1.1.26. Let $A_i, i \in I$ be an infinite set of rings, such that $A_i \neq 0$. Is there a ring A such that there is a homeomorphism of topological spaces as follows:
$$\operatorname{Spec} A = \bigsqcup_i \operatorname{Spec} A_i \text{ ?}$$

Exercise 1.1.27. Let $C = \operatorname{Spec} \mathbf{Z}[t,u]/tu$. Show that $C_1 := \operatorname{Spec} \mathbf{Z}[u]$ and $C_2 := \operatorname{Spec} \mathbf{Z}[t]$ are the irreducible components of C.

Exercise 1.1.28. Let $q = p^r$ be a prime power, and $\mathbf{F}_q$ a field with q elements. Recall that for any $\mathbf{F}_q$-algebra A, the *Frobenius* (or, in certain situations also referred to as the *absolute Frobenius*) is the ring homomorphism (!)
$$\mathrm{Frob} : A \to A, a \mapsto a^q.$$

The induced map on affine spectra is again denoted Frob:

$$\mathrm{Frob} : \operatorname{Spec} A \to \operatorname{Spec} A.$$

Prove that this latter map Frob is the *identity* on the level of the underlying sets(!)

In particular, it is a homeomorphism (on the level of the underlying topological spaces). Of course, Frob is *not* an isomorphism of rings for example for $A = \mathbf{F}_q[t]$.

Exercise 1.1.29. Let $A = \mathbf{F}_q[t_1, \dots, t_n]/(f_1, \dots, f_m)$ be a finitely generated $\mathbf{F}_q$-algebra. Fix an algebraic closure $\overline{\mathbf{F}}_q$ and write $\overline{A} := A \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$. For $\lambda_i \in \overline{\mathbf{F}}_q$, $i = 1, \dots, n$ such that $f_j(\lambda_1, \dots, \lambda_n) = 0$ (for all j), we consider the maximal ideal $\mathfrak{m} = (t_i - \lambda_i)\overline{A}$. (According to Hilbert's Nullstellensatz, to be proved below in Corollary 1.9.2, all maximal ideals of $\overline{A}$ are of that form.)

Consider the map

$$\mathrm{Frob}_A \otimes \mathrm{id}_{\overline{\mathbf{F}}_q} : \overline{A} \to \overline{A}.$$

Prove that the induced map on spectra

$$\operatorname{Spec} \overline{A} \to \operatorname{Spec} \overline{A}$$

has the property that it sends

$$\mathfrak{m}_{(\lambda_1, \dots, \lambda_n)} \mapsto \mathfrak{m}_{(\lambda_1^q, \dots, \lambda_n^q)}.$$

In other words, this is the map that geometrically corresponds to raising the coordinates to the q-th power. This map is of paramount importance in the study of algebraic geometry over a field of positive characteristic.

1.2 Local rings

Definition and Lemma 1.2.1. The following properties are equivalent:

(1) A has exactly one maximal ideal (which is commonly denoted $\mathfrak{m}$ or $\mathfrak{m}_A$), i.e., $\operatorname{Spec} A$ has exactly one closed point.

(2) $A \neq 0$ and $A \backslash A^\times$ (the elements in A that are not a unit) forms an ideal.

(3) $A \neq 0$ and if $f + g \in A^\times$ then $f \in A^\times$ or $g \in A^\times$.

(4) $A \neq 0$ and for any $f \in A$ we have $f \in A^\times$ or $1 - f \in A^\times$.

If these conditions are satisfied, A is called a *local ring*.

Proof. The simple proof is omitted. We only note that $A \backslash A^\times$ is the unique maximal ideal of A in this case. See, e.g., [Stacks, Tag 07BJ] for further details. □

Example 1.2.2.

- A field k is a local ring.
- $\mathbf{Z}$ is not a local ring (the maximal ideals are the principal ideals (p) for the prime numbers p).
- $k[t]$ is not a local ring since neither t nor $1-t$ is a unit in $k[t]$. Moreover, the maximal ideals are precisely the ones of the form (f), where f is an irreducible polynomial. (Immediately, there is more than one of them, namely $f = t$ and $f = t+1$. In fact there are infinitely many, even if k is a finite field.

Definition and Lemma 1.2.3. Let $f : A \to B$ be a ring homomorphism between two local rings. The following are equivalent:

(1) $f^{-1}(B^\times) = A^\times$. A ring homomorphism with that property is called *conservative*.

(2) $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$,

(3) $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.
The map f is called a *local map* in this event.

Proof. The proof is again very simple, since $\mathfrak{m}_B = B \backslash B^\times$. □

Recall that for any ring A and any prime ideal $\mathfrak{p} \subset A$, the *localization* is defined as

$$A_\mathfrak{p} := A[(A \backslash \mathfrak{p})^{-1}].$$

This is a local ring whose unique maximal ideal is $\mathfrak{p}A_\mathfrak{p}$. The quotient field

$$k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$$

of that maximal ideal is called the *residue field* of $\mathfrak{p}$. For each prime ideal $\mathfrak{p}$, there is a natural map

$$A \to A_\mathfrak{p} \to k(\mathfrak{p}). \tag{1.2.4}$$

This allows us to think of an element $f \in A$ as a function taking values in the residue fields (which are generally different for different $\mathfrak{p}$). The map $A_\mathfrak{p} \to A/\mathfrak{p}$ in (1.2.4) is a local map.

For a subset $Y \subset \operatorname{Spec} A$, we have

$$I(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p} = \ker \left(A \to \prod_{\mathfrak{p} \in Y} k(\mathfrak{p}) \right), \tag{1.2.5}$$

(if $\frac{a}{1} \in \mathfrak{p}A_\mathfrak{p}$, i.e. $\frac{a}{1} = \frac{p}{s}$ with $p \in \mathfrak{p}$ and $s \notin \mathfrak{p}$ then $ast = pt$ for some $t \notin \mathfrak{p}$, i.e., $ast \in \mathfrak{p}$, so that $a \in \mathfrak{p}$).

In topology, one understands a continuous map $f : X \to Y$ (to a certain extent) if one understands the so-called *fibers* $f^{-1}(y)$ for all $y \in Y$. Here is the corresponding algebro-geometric notion.

Lemma 1.2.6. Let $f : A \to B$ be a ring map and $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ the induced map on spectra. Fix some $\mathfrak{p} \subset A$. Then there is a homeomorphism

$$\operatorname{Spec}(B \otimes_A k(\mathfrak{p})) \xrightarrow{\cong} \varphi^{-1}(\mathfrak{p})$$

(where at the left the tensor product is formed using the canonical map (1.2.4) and the space at the right carries the subspace topology of $\operatorname{Spec} B$).

Proof. We have

$$B \otimes_A A_\mathfrak{p} = B \otimes_A A[(A \backslash \mathfrak{p})^{-1}] = B[f(A \backslash \mathfrak{p})^{-1}]$$

and by Exercise 1.1.22(1) the spectrum of this ring is homeomorphic to $\{\mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{q} \cap f(A \backslash \mathfrak{p}) = \varnothing\}$. That latter condition is equivalent to $f^{-1}(\mathfrak{q}) \cap (A \backslash \mathfrak{p}) = \varnothing$, or to $f^{-1}(\mathfrak{q}) \subset \mathfrak{p}$.

We then have

$$B \otimes_A k(\mathfrak{p}) = B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}.$$

According to Exercise 1.1.22(2), its spectrum is homeomorphic to $\{\mathfrak{q} \in \operatorname{Spec} B \otimes_A A_\mathfrak{p} \mid f(\mathfrak{p}A_\mathfrak{p}) \subset \mathfrak{q}\}$, which is equivalent to $f(\mathfrak{p}) \subset \mathfrak{q}$ and, in the presence of the above condition, to $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. □

Corollary 1.2.7. In the above situation, we have

$$\varphi(\operatorname{Spec} B) = \{\mathfrak{p} \in \operatorname{Spec} A \mid B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \neq 0\}.$$

Exercises

Exercise 1.2.8. Let A be a ring and $X = \operatorname{Spec} A$. Show that A is local if and only if for every open covering $X = \bigcup_{i \in I} U_i$, there is some i such that $U_i = X$. One refers to this statement by saying that (spectra of) local rings are the *points* of the Zariski topology.

Exercise 1.2.9. Show that the assignment

$$\mathfrak{p} \mapsto (A \stackrel{(1.2.4)}{\to} k(\mathfrak{p}))$$

gives rise to a bijection

$$\operatorname{Spec} A \stackrel{1:1}{\to} \bigsqcup_k \operatorname{Hom}_{\mathrm{Rings}}(A, k)/\sim,$$

where at the right the disjoint union runs over the collection (actually, a proper class) of all fields and a ring homomorphism $f : A \to k$ is identified with $g : A \to k'$ (possibly for some other field k') if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & k \arrow[d] \\ & k'. \end{tikzcd}$$

Exercise 1.2.10. (Solution at p. 106) Let $\pi : \mathbf{A}^1 := \operatorname{Spec} \mathbf{Z}[t] \to \operatorname{Spec} \mathbf{Z}$ be the map induced by the inclusion $\mathbf{Z} \subset \mathbf{Z}[t]$. Using the description of the points of $x \in \mathbf{A}^1$ from Exercise 1.3.7, describe $\pi(x)$ for each point $x \in \mathbf{A}^1$.

1.3 Dimension

Definition 1.3.1. The *Krull dimension* of a topological space X, is

$$\dim X := \sup\{n | Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n (\subset X)\},$$

where the supremum is taken over all chains of irreducible closed subsets. (Thus $\dim X = \infty$ iff arbitrarily long chains exist, it is by convention $-\infty$ iff $X = \varnothing$.)

Definition 1.3.2. The *Krull dimension* of A is

$$\dim A := \dim \operatorname{Spec} A = \sup\{n | \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n\},$$

where the supremum is taken over all chains of prime ideals.

Example 1.3.3. (1) A ring A is zero-dimensional iff every prime ideal is maximal. This is the case if A is a field or, more generally, if A is an Artinian ring (see [Stacks, Tag 00JA]; in fact the Artinian rings are precisely the zero-dimensional Noetherian rings [Stacks, Tag 00KH]). In particular, if A is a k-algebra that is finitely generated as a k-vector space (as opposed to being finitely generated as a k-algebra!), we have $\dim A = 0$. We will give a complete description of reduced 0-dimensional rings in Lemma 1.4.9.

(2) For a principal ideal domain A that is not a field, such as $A = \mathbf{Z}$ or $A = k[t]$, we have $\dim A = 1$: the chain of prime ideals are $(0) \subsetneq (f)$ (with f being irreducible elements of A). Note that (0) is the generic point, and the ideals (f) are the closed points.

(3) For $A = k[t_1, \ldots, t_n]$ we have the chain of prime ideals

$$(0) \subset (t_1) \subset (t_1, t_2) \subset \cdots \subset (t_1, \ldots, t_n),$$

so $\dim A \geqslant n$. We will see below that there are no longer chains of prime ideals, i.e., $\dim k[t_1, \ldots, t_n] = n$ (Corollary 1.9.3).

(4) We have $\dim \mathbf{Z}[t] = 2$, cf. Exercise 1.3.7.

(5) By definition, the dimension only "sees" the underlying topological space of $\operatorname{Spec} A$. In particular, we have $\dim A = \dim A_{\mathrm{red}}$, cf. (1.1.23).

Despite its simplicity, the dimension of a ring is actually not always harmless to work with. We will use the following properties, which are non-trivial to prove.

Theorem 1.3.4. Let A be a Noetherian local ring, with $\mathfrak{m}$ its maximal ideal.

(1) [Stacks, Tag 00KD] If $\mathfrak{m}$ is generated by, say, n elements, then

$$\dim A \leqslant n.$$

In particular,

$$\dim A < \infty.$$

(2) [Stacks, Tag 00KW] For $x \in \mathfrak{m}$ we have

$$\dim A/x \geqslant \dim A - 1.$$

If x not contained in a minimal prime ideal of A (for example x a non-zero divisor) we have

$$\dim A/x = \dim A - 1.$$

(I.e., modding out x causes the dimension to drop at most by 1, and it does drop if x is not a zero-divisor.)

For a Noetherian, but non-local ring, we may have $\dim A = \infty$ (despite all its localizations having finite dimension), see [Stacks, Tag 02JC] for an example of the form $A = k[x_1, x_2, \dots][S^{-1}]$.

Theorem 1.3.5. For a Noetherian ring A,

$$\dim A[t] = \dim A + 1.$$

(I.e., one side is finite iff the other is, and equality holds in that case.)

This theorem is originally due to Krull. The theorem also holds if A is a valuation ring. See [Bre+73] for a uniform proof of both statements.

Warning 1.3.6. For an arbitrary ring, a theorem due to Seidenberg[1] asserts that if $\dim A = d$, then

$$d + 1 \leqslant \dim A[t] \leqslant 2d + 1,$$

and (for appropriate non-Noetherian rings), each value in between $d+1$ and $2d+1$ can be attained. See [Bou06, Chapter VIII, §2, Corollaire 2]. Because of that, we will consider the dimension of (local) rings only in the context of Noetherian rings in this course. As an outlook, we just mention the existence of *valuative dimension* of a ring, introduced by Jaffard [Jaf60] and denoted $\operatorname{dimv} A$. See, e.g., [WK24, §5.4.3] for a textbook account. It has the following properties:

- One has $\dim A \leqslant \operatorname{dimv} A$.
- If A is Noetherian or a valuation ring, then $\dim A = \operatorname{dimv} A$.
- For any ring A, we have $\operatorname{dimv} A[t] = \operatorname{dimv} A$.

[1] https://msp.org/pjm/1954/4-4/pjm-v4-n4-p09-p.pdf

Exercises

Exercise 1.3.7. (Solution at p. 106)
(1) Prove that the prime ideals of $\mathbf{Z}[t]$ are precisely the following:

- The zero ideal (0).
- A principal ideal of the form $p\mathbf{Z}[t]$, for a prime number p.
- A principal ideal of the form (f), where $f \in \mathbf{Z}[t]$ is an irreducible polynomial of degree > 0 whose coefficients have no common prime divisor (equivalently, the ideal $I \subset \mathbf{Z}$ generated by the coefficients of f satisfies $I = \mathbf{Z}$).
- An ideal of the form (p, f), where p is a prime, f is again an irreducible polynomial such that its image in $\mathbf{F}_p[t]$ is still irreducible.

Hint: any ideal is of the form $(a_1, \dots, a_n, f_1, \dots, f_m)$ with $a_i \in \mathbf{Z}$ and f_j polynomials of positive degree. The above four cases correspond to $m, n = 0, 1$ respectively.

(2) Prove that the first is the generic point, the latter type of ideals the maximal ideals. Deduce

$$\dim \mathbf{Z}[t] = 2.$$

For this reason, on refers to $\mathbf{A}^1_{\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[t]$ as an *arithmetic surface*.

(3) Discuss the relation of this fact with $\dim \mathbf{F}_p[t] = 1$ and $\dim \mathbf{Q}[t] = 1$.

Exercise 1.3.8. (Solution at p. 106)
(1) Give an example of a ring A with 5 prime ideals, of which 4 are maximal and 1 is not maximal.
(2) What is the dimension of A?

1.4 Flatness

Recall that for a ring A, an A-module M is called *flat* (or *flat over* A to emphasize the ring A) if

$$M \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_A$$

is an *exact* functor. (For any M, this functor is right exact, so the actual condition is that $M \otimes_A N$ is a submodule of $M \otimes_A N'$, for any submodule $N \subset N'$.) If $f : A \to B$ is a ring homomorphism, we say that f is *flat* (or that B is *flat over* A) if B is flat as an A-module.

A flat module (or algebra) is called *faithfully flat* if $M \otimes_A -$ is conservative (i.e., $M \otimes_A N \to M \otimes_A N'$ is an isomorphism (if and) only if $N \to N'$ is an isomorphism); equivalenty M is flat and $M \otimes_A N = 0$ (if and) only if $N = 0$.

Example 1.4.1. (1) Any free A-module $M \cong \bigoplus_{i \in I} A$ is flat (and it is faithfully flat iff $M \neq 0$ or equivalently $I \neq \emptyset$). Indeed, $M \otimes_t hAN = \bigoplus_{i \in I} N$.

- As a special case: $A[t]$ and more generally $A[t_i, i \in I]$ is a faithfully flat A-algebra.
- Another special case: if $A = k$ is a field, *any* k-module is free (i.e., has a basis), and in particular any module or algebra over a field k is flat.

(2) Unlike the inclusion $A \subset A[t]$, the unique ring homomorphism $A[t] \to A$ satisfying $t \mapsto 0$ is *not* flat: the multiplication by $t : A[t] \to A[t]$ is injective, but after applying $A \otimes_{A[t]} -$, i.e., applying $(-)/t$, we get the map $t = 0 : A = A[t]/t \to A$, which is no longer injective. We will see in Proposition 1.8.7 that any flat ring map $A \to B$ often induce *open maps* $\operatorname{Spec} B \to \operatorname{Spec} A$ (i.e., images of open subsets are open). However, the image of $\operatorname{Spec} A \to \operatorname{Spec} A[t]$ is *closed.*

(3) Filtered colimits of flat A-modules are flat. This is true since $\operatorname{colim} M_i \otimes_A - = \operatorname{colim}(M_i \otimes_A -)$ and taking a filtered colimits of exact sequences of A-modules gives an exact sequence.

In fact, this is not just an example, but all flat modules arise this way. More precisely, *Lazard's theorem* asserts that M is flat iff it is a filtered colimit of finite free A-modules [Stacks, Tag 058G].

(4) As a special case of filtered colimits, we have that any localization $A[f^{-1}]$ or $A[S^{-1}]$ of a ring is flat. (There is an isomorphism of A-modules $A[f^{-1}] = \operatorname{colim}(A \xrightarrow{f} A \xrightarrow{f} \dots.)$ It is typically not *faithfully* flat though: we have $A[f^{-1}] \otimes_A A/f = (A/f)[f^{-1}] = 0$, but $A/f \neq 0$ (unless f is a unit).

(5) If $A \to B$ is a ring homomorphism and M is a flat A-module, then $M \otimes_A B$ is a flat B-module. Indeed

$$M \otimes_A B \otimes_B - = M \otimes_A -$$

is an exact functor. This property is referred to by saying that "flatness is preserved under *base change*".

Lemma 1.4.2. Suppose M is a flat A-module. The following are equivalent:

(1) M is faithfully flat,

(2) $M/IM (= A/I \otimes_A M) \neq 0$ for any ideal $I \subsetneq A$,

(3) $M/\mathfrak{p}M (= A/\mathfrak{p} \otimes_A M) \neq 0$ for any prime ideal $\mathfrak{p} \subset A$,

(4) $M/\mathfrak{m}M (= A/\mathfrak{m} \otimes_A M) \neq 0$ for any maximal ideal $\mathfrak{m} \subset A$.

Proof. Clearly we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(2) $\Rightarrow$ (1): suppose N is an A-module such that $M \otimes_A N = 0$. Any element $n \in N$ gives rise to an exact sequence

$$0 \to I := \operatorname{Ann}_N n \to A \xrightarrow{n} N,$$

which (using that M is flat!) then gives

$$0 \to I \otimes_A M \to M \to N \otimes_A M.$$

By assumption the right hand term vanishes, so that the left map is an isomorphism, which by (2) implies $I = A$, i.e., $n = 0$.

(4) $\Rightarrow$ (2): any proper ideal I is contained in some maximal ideal $\mathfrak{m}$, i.e., there is a surjection $A/I \to A/\mathfrak{m}$. Applying $- \otimes_A M$ gives a surjection $M/IM \to M/\mathfrak{m}M$. Thus, if $M/\mathfrak{m}M \neq 0$, then also $M/IM \neq 0$. □

Lemma 1.4.3. Let $f : A \to B$ be a flat ring map.

(1) Then f is faithfully flat if and only if f induces a *surjective* map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$.

(2) If f is a (flat) local map of local rings, it is faithfully flat.

Proof. (1): Let $\mathfrak{p} \subset A$ be a prime ideal. Then $\varphi^{-1}(\mathfrak{p}) = \operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ (Lemma 1.2.6). This is empty iff $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = (B/\mathfrak{p}B)_\mathfrak{p} = 0$ iff $B/\mathfrak{p}B = 0$. (For the latter equivalence we use that for a ring C and a multiplicatively closed subset $S \subset C$, $0 \notin S$ we have $C = 0$ iff $C[S^{-1}] = 0$.) We conclude by Lemma 1.4.2.

(2): since the map is local, we have $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$, so that we have a natural map $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B \to B/\mathfrak{m}_B$, which is a ring homomorphism between fields and therefore injective; in particular $B/\mathfrak{m}_A B \neq 0$. □

Example 1.4.4. Let $f_1, \dots, f_n \in A$. The following are equivalent:

(1) The A-algebra
$$\prod_{i=1}^{n} A[f_i^{-1}]$$
is faithfully flat.

(2) The f_i generate the unit ideal.

(3) $\operatorname{Spec} A = \bigcup_{i=1}^{n} D(f_i) = \bigcup_{i=1}^{n} \operatorname{Spec} A[f_i^{-1}]$.

Indeed, the equivalence (2) $\Leftrightarrow$ (3) was already shown in Lemma 1.1.10.

Corollary 1.4.5. An A-module M is flat iff all the localizations $M_{\mathfrak{p}}$ (for all prime ideals $\mathfrak{p} \subset A$) are flat over $A_{\mathfrak{p}}$.

Proof. Note that $M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$ for any M and A. Thus the direction "$\Rightarrow$" holds by stability of flatness under base change (Example 1.4.1(5)).

The direction "$\Leftarrow$" holds since the map $A \to \prod_{\mathfrak{p}} A_{\mathfrak{p}}$ is *faithfully flat* by Lemma 1.4.3: given an injection $N \to N'$ of A-modules, the injectivity of $M \otimes_A N \to M \otimes_A N'$ can be checked after applying $- \otimes_A A_{\mathfrak{p}}$ (for all $\mathfrak{p}$). $\square$

Having these preliminary technical properties of flatness at our disposal, we now come to a key reason why flatness is important in commutative algebra and algebraic geometry.

Lemma 1.4.6. If B is a faithfully flat A-algebra, and M is an A-module, then there is an exact equence (of A-modules), called the *Amitsur complex*
$$0 \to M \to B \otimes_A M \to (B \otimes_A B) \otimes_A M$$
where the first map is $m \mapsto 1 \otimes m$ and the second map is given by $b \otimes m \mapsto b \otimes 1 \otimes m - 1 \otimes b \otimes m$. (The key case to consider is $M = A$, in which case this simplifies to
$$0 \to A \xrightarrow{f} B \stackrel{f\otimes 1 - 1\otimes f}{\to} B \otimes_A B.) \tag{1.4.7}$$

Proof. For simplicity of notation, we spell out the proof for $M = A$; in general one simply appends the functor $- \otimes_A M$ to all the arguments in the proof below.

(1) We first prove it if f admits a section, i.e., an algebra map $s : B \to A$ such that $s \circ f = \mathrm{id}$. Clearly the map f is injective then, so the sequence (1.4.7) is exact at the left. To check the exactness in the middle, consider the map $k : B \otimes_A B \to B$ satisfying $k(b \otimes b') = bfs(b')$. It satisfies $k(b \otimes 1 - 1 \otimes b) = b - fs(b)$, so if $b \otimes 1 - 1 \otimes b = 0$, then $b = fs(b) \in f(A)$. (To demystify what might look an unmotivated trick above see Exercise 1.5.9.)

(2) Suppose $A \to A'$ is faithfully flat. Then we prove the claim for $A' \to A' \otimes_A B$ implies the one for $A \to B$. Indeed, the complex (1.4.7) for $A' \to A' \otimes_A B$ is obtained from the one for $A \to B$ by applying $- \otimes_A A'$ and this functor is exact (by flatness) and conservative (by faithfulness).

(3) The map $\mathrm{id} \otimes 1 : B \to B \otimes_A B$ arises from f by applying $B \otimes_A -$. In addition, the map $\mathrm{id} \otimes 1$ has a section given by the multiplication, so the asserted exactness holds by the first step. By the second step, it then holds for f. $\square$

Definition 1.4.8. A ring A is called *absolutely flat* if *any* A-module M is flat.

We have noted above that any field is absolutely flat. Most rings are *not* absolutely flat, for example $\mathbf{Z}$ is not absolutely flat since $\mathbf{Z}/p$ is not a flat $\mathbf{Z}$-module. The following characterization of absolutely flat rings is due to Olivier [Oli78]. Part (2) appears in [BSY22, Proposition 4.41]. We will later use this notion to prove Chevalley's theorem on images of constructible sets.

Lemma 1.4.9. The following are equivalent:

(1) A is absolutely flat.

(2) Any ring homomorphism $\mathbf{Z}[t] \to A$ factors uniquely as shown below:

$$\begin{tikzcd} \mathbf{Z}[t] \arrow[r, "t\mapsto a"] \arrow[d, "t\mapsto(0,t)"'] & A \\ \mathbf{Z}\times\mathbf{Z}[t^{\pm 1}] \arrow[ur, dotted, "\exists!"'] \end{tikzcd} \tag{1.4.10}$$

(3) Any $a \in A$ can be written as

$$a = eu$$

for an idempotent e and a unit u.

(4) A is reduced and $\operatorname{Spec} A$ is Hausdorff.

(5) A is reduced and satisfies $\dim A = 0$.

(6) All its local rings $A_{\mathfrak{p}}$ are fields.

Proof. Independently of the property of being absolutely flat etc., recall the category-theoretic fact that (2) admits a unique lift if the right hand-vertical map h in the pushout diagram below is an isomorphism:

$$\begin{tikzcd} \mathbf{Z}[t] \arrow[r, "t\mapsto a"] \arrow[d] & A \arrow[d, "h"] \\ \mathbf{Z}\times\mathbf{Z}[t^{\pm 1}] \arrow[r] & A\otimes_{\mathbf{Z}[t]}(\mathbf{Z}\times\mathbf{Z}[t^{\pm 1}]) \end{tikzcd} \tag{1.4.11}$$

In more plain terms, this just means that the following map is a ring isomorphism:

$$A \to A/a \times A[a^{-1}], x \mapsto (x, \frac{x}{1}). \tag{1.4.12}$$

We prove (1) $\Rightarrow$ (2) by showing that h is an isomorphism provided that A is absolutely flat. Since A is absolutely flat, the map h is flat. It is also surjective on the level of spectra (in fact a *bijection* $V(a) \sqcup D(a) \to \operatorname{Spec} A$) and therefore faithfully flat. By faithful flatness we have an *exact* Amitsur complex (1.4.7), cf. Lemma 1.4.6:

$$0 \to A \xrightarrow{h} B \xrightarrow{b\mapsto b\otimes 1 - 1\otimes b} B\otimes_A B.$$

This complex arises from the similar complex for $g : \mathbf{Z}[t] \to \mathbf{Z}\times\mathbf{Z}[t^{\pm 1}]$ by applying $A\otimes_{\mathbf{Z}[t]} -$. The (non-exact) Amitsur complex for the (non-flat) map g is

$$0 \to \mathbf{Z}[t] \to \mathbf{Z}\times\mathbf{Z}[t^{\pm 1}] \to (\mathbf{Z}\times\mathbf{Z}[t^{\pm 1}])\otimes_{\mathbf{Z}[t]}(\mathbf{Z}\times\mathbf{Z}[t^{\pm 1}]).$$

We claim the right hand map is zero. Indeed, $\mathbf{Z}\otimes_{\mathbf{Z}[t]}\mathbf{Z}[t^{\pm 1}] = 0$ (geometrically this corresponds to the fact that $0 \cap \mathbf{G}_{\mathrm{m}} = \varnothing$), so we only have to consider

$$\mathbf{Z} \to \mathbf{Z}\otimes_{\mathbf{Z}[t]}\mathbf{Z},$$

$$\mathbf{Z}[t^{\pm 1}] \to \mathbf{Z}[t^{\pm 1}]\otimes_{\mathbf{Z}[t]}\mathbf{Z}[t^{\pm 1}].$$

In both cases, for an element f in the domain, we have $f\otimes 1 = 1\otimes f$ in the target, so the map $f \mapsto f\otimes 1 - 1\otimes f$ is zero in the Amitsur complex for g and hence this is also the case for the Amitsur complex for h. Hence h is an isomorphism, confirming (2).

(2) $\Leftrightarrow$ (3) is left as Exercise 1.4.17.

(3) ⇒ (4): suppose $a^n = 0$ for some $a \in A$. Writing $a = eu$ with an idempotent e and a unit u, we have $a^n = e^n u^n = 0$, so $0 = e^n = e^{n-1} = \cdots = e$, so that $a = 0$. We prove $\operatorname{Spec} A$ is Hausdorff: consider two distinct prime ideals, say $\mathfrak{p} \subsetneq \mathfrak{q}$ (i.e., $\mathfrak{p} \rightsquigarrow \mathfrak{q}$). Pick an element $a \in \mathfrak{q} \backslash \mathfrak{p}$. Again, writing $a = eu$ as before, we have $e = u^{-1} a \in \mathfrak{q}$, but $e \notin \mathfrak{p}$. For any idempotent e, we have a decomposition into two clopen (closed and open) subsets $\operatorname{Spec} A = V(e) \sqcup D(e)$. (Indeed, for an idempotent e, one has $V(1-e) = D(e)$, as one checks readily. Another approach to seeing this is offered by Exercise 1.5.7.) This gives two open subsets separating $\mathfrak{p}$ and $\mathfrak{q}$.

(4) ⇒ (5): if we had $\mathfrak{p} \subsetneq \mathfrak{q}$ then $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$, so $\mathfrak{p}$ and $\mathfrak{q}$ could not be separated by two open subsets.

(5) ⇒ (6): For $\mathfrak{p} \in \operatorname{Spec} A$, $\operatorname{Spec} A_\mathfrak{p} = \{\mathfrak{p}\}$. Like A, $A_\mathfrak{p}$ is reduced. Such rings are fields.

(6) ⇒ (1): An A-module M is flat iff all the $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$ (Corollary 1.4.5), and fields are absolutely flat. □

Remark 1.4.13. As a forecast to the upcoming notion of morphisms of affine schemes, condition (2) above is equivalent to the existence and unicity of a map of affine schemes as pictured below (where at the right we have the map $\operatorname{Spec} \mathbf{Z} \to \mathbf{A}^1$ induced by $\mathbf{Z}[t] \to \mathbf{Z}$, $t \mapsto 0$), and the standard inclusion (1.6.33)):

$$\begin{tikzcd} & 0 \sqcup \mathbf{G}_\mathrm{m} \arrow[d] \\ \operatorname{Spec} A \arrow[ur, dotted, "\exists!"] \arrow[r, "a"] & \mathbf{A}^1. \end{tikzcd} \tag{1.4.14}$$

Exercises

Exercise 1.4.15. If B is as in Example 1.4.4, prove the exactness of the Amitsur complex

$$0 \to A \to \prod_{i=1}^n A[f_i^{-1}] \to \prod_{i,j=1}^n A[f_i^{-1} f_j^{-1}]$$

by hand (cf. Lemma 1.4.6). Explain how the complex fails to be exact if the f_i do not generate the unit ideal in A.

Exercise 1.4.16. Suppose $f : A \to B$ is faithfully flat and an epimorphism (in the category of rings, i.e., for $B \underset{g_1}{\overset{g_2}{\rightrightarrows}} C$ with $g_1 f = g_2 f$, we have $g_1 = g_2$.) Show that f is an isomorphism.

Give an example of a (non-faithfully) flat epimorphism that is not an isomorphism.

Exercise 1.4.17. (Solution at p. 106)

(1) Prove the equivalence of (3) and (2) in Lemma 1.4.9.

Hint: recall or prove that there is a bijection $\operatorname{Hom}_{\mathrm{Rings}}(\mathbf{Z}[t], A) = A$ (given by $f \mapsto f(t)$). Establish a related description of $\operatorname{Hom}_{\mathrm{Rings}}(\mathbf{Z} \times \mathbf{Z}[t], A)$ and then $\operatorname{Hom}_{\mathrm{Rings}}(\mathbf{Z} \times \mathbf{Z}[t^{\pm 1}], A)$.

(2) Also prove that the map $\mathbf{Z}[t] \to \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}]$ is an epimorphism (in the category of rings). Conclude that in (2) one may equivalently drop the unicity of the lift (and only demand its existence).

1.5 The structural sheaf on $\operatorname{Spec} A$

As a mere *set* and also as a *topological space*, the spectrum of a ring does not distinguish between a ring A and its associated reduced ring $A_{\mathrm{red}} := A/\sqrt{0}$ (Exercise 1.1.22). Possibly even more dramatically, the spectrum does not distinguish between fields: $\operatorname{Spec} \mathbf{F}_p$ and $\operatorname{Spec} \mathbf{Q}$ are both the one-point topological space. Thus, one needs to refine this topological space $\operatorname{Spec} A$ with an additional datum, namely the structural sheaf. The purpose of the structural sheaf is to record the "allowed" functions on the open subsets of $\operatorname{Spec} A$. This will in particular allow to recover the ring A.

We begin by recalling some basic notions pertaining to sheaves. See, e.g., [Har83, §II.1], [GW20, §II] or [Bre97, §§I.1–3] for more in-depth textbook accounts. A *presheaf* F on a topological space X is a functor

$$F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Set}.$$

I.e., for any open $U \subset X$ there is a set $F(U)$ and whenever $U \subset V$ are two open subsets of X, there is a map (often called *restriction map*) $\mathrm{res}^U_V : F(V) \to F(U)$ that is compatible with further restriction maps in the obvious sense. Note that the restriction map goes in the "wrong" way, i.e., F is a *contravariant* functor.

A morphism of presheaves $F \to G$ is just a natural transformation of functors. Equivalently, it is a collection of maps $F(U) \to G(U)$ that commute with the restriction maps of F and G. This defines the category $\mathrm{PSh}(X)$ of presheaves. More succinctly, $\mathrm{PSh}(X) := \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Set})$.

Of particular importance is the functor, called the *global sections* functor

$$\Gamma : \mathrm{PSh}(X) \to \mathrm{Set}, F \mapsto \Gamma(X, F) := \Gamma(F) := F(X).$$

A presheaf is called a *sheaf* if for any open subset $U \subset X$ and any covering $U = \bigcup_{i \in I} U_i$, the natural map

$$F(U) \stackrel{\prod \mathrm{res}^U_{U_i}}{\to} \mathrm{eq}\left(\prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j) \right)$$

is an isomorphism. (Here the two maps in the equalizer are the following: given a collection $(f_i)_i$ with $f_i \in F(U_i)$, its image in the (i,j)-component in the right hand product is a) $\mathrm{res}^{U_i}_{U_i \cap U_j}(f_i)$, respectively b) $\mathrm{res}^{U_j}_{U_i \cap U_j}(f_i)$. In other words, the equalizer consists of those collections (f_i) that restrict to the same elements in $F(U_i \cap U_j)$. A morphism of sheaves is, by definition, just a morphism of presheaves. In other words, we define the category $\mathrm{Shv}(X)$ of sheaves to be the full subcategory of $\mathrm{PSh}(X)$ consisting of the presheaves satisfying the above condition.

A noteworthy consequence of the definition is that

$$F(\varnothing) = \{*\}$$

for any sheaf, by applying the sheaf condition to the covering of $\varnothing$ consisting of *no* open sets, i.e., $\prod_\varnothing = \{*\}$.

A (pre)sheaf of abelian groups or rings is defined similarly, i.e., $F(U)$ are abelian groups (or rings) and the restriction maps are group (or ring) homomorphisms. Note that for abelian groups the sheaf condition can be rephrased as saying that the sequence

$$0 \to F(U) \to \prod_i F(U_i) \stackrel{\mathrm{res}^{U_i}_{U_i \cap U_j} - \mathrm{res}^{U_j}_{U_i \cap U_j}}{\to} \prod_{i,j} F(U_i \cap U_j) \tag{1.5.1}$$

is an exact sequence.

A typical example of a sheaf (on some fixed topological space X) is given by

$$F(U) := \{f : U \to \mathbf{R} \text{ continuous}\}.$$

Indeed, this is a sheaf (actually a sheaf of rings) since a collection of continuous functions on U_i can be glued to a function on $U = \bigcup_i U_i$ precisely if $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Similarly, if, say $X = \mathbf{R}^n$ or if X is a differentiable manifold, one may consider the sheaf $F(U) := \{f : U \to \mathbf{R} \text{ differentiable}\}$. The structural sheaf $\mathcal{O}$ on $\operatorname{Spec} A$ constructed below is similar in spirit, except that instead of continuous or differentiable functions, we consider functions that are–in a sense made precise by (1.5.4)–algebraic.

The structural sheaf $\mathcal{O}$ and most of the sheaves (and morphisms of sheaves) we will consider actually arise by knowing what they do on a basis of the topology, by means of the following lemma.

Lemma 1.5.2. Let X be a topological space. Let $\mathcal{B}$ be a basis of the topology on X. Then restricting a sheaf to its restriction on $\mathcal{B}$ yields an equivalence of categories

$$\mathrm{Shv}(X) \xrightarrow{\cong} \mathrm{Shv}(\mathcal{B}, \mathrm{Set}),$$

where $\mathrm{Shv}(\mathcal{B}, \mathrm{Set}) \subset \mathrm{PSh}(\mathcal{B}, \mathrm{Set}) := \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Set})$ denotes the full subcategory consisting of those presheaves satisfying the sheaf condition for any $U = \bigcup U_i$ where all U_i and U are elements of $\mathcal{B}$.

Proof. Clearly, restricting a sheaf to $\mathcal{B} \subset \mathrm{Open}(X)$ gives a sheaf on $\mathcal{B}$. Conversely, given some $F \in \mathrm{Shv}(\mathcal{B}, \mathrm{Set})$, define a sheaf G by declaring for each open $V \subset X$:

$$G(V) := \{(f_U)_{U \subset V, U \in \mathcal{B}} \in G(U) \mid \mathrm{res}_U^{U'}(f_{U'}) = f_U\}.$$

(More succinctly, $G(V) = \lim(\{U \in \mathcal{B}, U \subset V\}^{\mathrm{op}} \xrightarrow{G} \mathrm{Set})$.) One checks that these two functors are inverse to each other essentially by repeatedly applying the sheaf condition, cf. [Stacks, Tag 009O]. □

Lemma 1.5.3. Let A be a ring. Then there is a unique sheaf, called the *structural sheaf* and denoted $\mathcal{O}_A$ or $\mathcal{O}$ on $\mathrm{Spec}\, A$ satisfying

$$\mathcal{O}_A(D(f)) = A[f^{-1}] \tag{1.5.4}$$

(and restriction maps given by the natural maps $A[f^{-1}] \to A[(fg)^{-1}]$ between localizations). In particular,

$$\mathcal{O}(\mathrm{Spec}\, A) = A.$$

More generally, if M is an A-module, there is a unique sheaf denoted $\widetilde{M}$ satisfying

$$\widetilde{M}(D(f)) = M[f^{-1}](= M \otimes_A A[f^{-1}]). \tag{1.5.5}$$

Proof. We define a presheaf on the basis $\mathcal{B}$ of fundamental opens by taking (1.5.5) as a definition. This is well-defined, i.e., independent of the choice of f, by (1.1.11). Also note that the restriction maps for $D(f) \subset D(g)$ exist by Lemma 1.1.10(3).

We prove that this defines a sheaf on $\mathcal{B}$ in two steps: First, we first check the sheaf condition if some $U = D(f)$ is covered by finitely many $U_i = D(f_i)$. For simplicity of the notation, we replace $\mathrm{Spec}\, A$ by $U = \mathrm{Spec}\, A[1/f]$, noting that $D(f_i)$ inside $\mathrm{Spec}\, A$ agrees with $D(f_i)$ inside $\mathrm{Spec}\, A[1/f]$. Thus, we are in the situation that $U_i = D(f_i)$ are a finite open covering of $\mathrm{Spec}\, A$. In this case the exactness of

$$0 \to M \to \prod_i M[f_i^{-1}] \to \prod_{i,j} M[(f_i f_j)^{-1}] \tag{1.5.6}$$

is exactly the content of Lemma 1.4.6, given that $M[r^{-1}] = M \otimes_A A[r^{-1}]$.

Second, we check that the sheaf condition also holds if $U = D(f)$ is covered by infinitely many $U_i = D(f_i)$, $i \in I$. The key here is that $U = \mathrm{Spec}\, A[f^{-1}]$ is quasi-compact (Lemma 1.1.10(4)). Thus, there is a finite subset $K \subset I$ such that the U_i for $i \in K$ already cover U. This directly implies the exactness of (1.5.6) at the left: if $m \in M$ is mapped to zero, it is in particular zero in all the components for $i \in K$, and thus $m = 0$ by the first step. Similarly, if $(m_i) \in \prod_i M[f_i^{-1}]$ is mapped to zero at the right, then for the above finite subset $K \subset I$, there is by the first step an element $m_K \in M$ such that $m_K|_{U_i} = m_i$ for all $i \in K$. Let $j \in I \backslash K$ be any element and put $J := K \cup \{j\}$. Again using the first step, there is an element $m_L \in M$ such that $m_L|_{U_i} = m_i$, this time for all $i \in L$. In particular, $m_L - m_K = 0 \in M[f_i^{-1}]$ for all $i \in K$. Using the exactness of (1.5.6) at the left, $m_L = m_K$ (as elements in M, and in particular also in $M[f_j^{-1}]$). Since j was arbitrary, we are done.

We finally apply Lemma 1.5.2 to obtain a sheaf defined on all open subsets of $\mathrm{Spec}\, A$. □

Exercises

Exercise 1.5.7. Recall that a subset of a topological space is called *clopen* if it is both open and closed. Also recall that an element $e \in A$ is called an *idempotent* if

$$e^2 = e.$$

Show that the mapping

$$\{e \in A, e \text{ idempotent}\} \to \{\text{clopen subsets } W \subset \operatorname{Spec} A\}, e \mapsto V(e)$$

is a bijection.

Hint: what does the sheaf axiom tell about W and its complement?

Exercise 1.5.8. A topological space X is called *connected* if whenever

$$X = U \sqcup V$$

(a disjoint union of two open subsets), one has $X = U$ or $X = V$.

- Prove that any irreducible space X is connected.
- Show that $\operatorname{Spec} \mathbf{Z}[t, u]/tu$ is connected (but not irreducible, cf. Exercise 1.1.27).
- Prove that $\operatorname{Spec} A$ is connected iff the only idempotents in A are 0 and 1.

Exercise 1.5.9. Let X be a topological space and F a presheaf on X. Show that the sheaf condition in (1.5) is automatically satisfied if $U_i = X$ for some i. Relate this observation to the first step in the proof of Lemma 1.4.6.

Exercise 1.5.10. The following statement is referred to by saying that "sheaves *glue*". Related statements concerning glueing maps of locally ringed spaces and glueing schemes are discussed in Exercise 1.6.30 and Lemma 2.1.5.

Let X be a topological space and $X = \bigcup_{i \in I} U_i$ a (possibly infinite) open covering. Write $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_i \cap U_j \cap U_k$ for $i, j, k \in I$. Let use be given:

- for each i, a sheaf $F_i \in \operatorname{Shv}(U_i)$,
- for each $i, j \in I$, isomorphisms of sheaves $\phi_{ij} : F_i|_{U_{ij}} \xrightarrow{\cong} F_j|_{U_{ij}}$

such that

$$\phi_{jk} \circ \phi_{ij} = \phi_{ij},$$

once we restrict these to U_{ijk}. (This condition is called the *cocycle condition*).

Construct a sheaf $F \in \operatorname{Shv}(X)$ and isomorphisms $F|_{U_i} \cong F_i$.

Hint: this can be deduced from Lemma 1.5.2. Note that applying the cocycle condition to $i = j = k$ implies $\phi_{ii} = \mathrm{id}$.

Remark 1.5.11. Along similar lines, one may observe that given another such collection F'_i, ϕ'_{ij}, and morphisms $f_i : F_i \to F'_i (\in \operatorname{Shv}(U_i))$ that are compatible with the ϕ_{ij}, ϕ'_{ij} there is naturally a morphism $F \to F'$. Yet more comprehensively, one can consider the natural restriction functor

$$\operatorname{Shv}(X) \to \lim \left(\prod_i \operatorname{Shv}(U_i) \rightrightarrows \prod_{i,j} \operatorname{Shv}(U_{ij}) \to \prod_{i,j,k} \operatorname{Shv}(U_{ijk}) \right),$$

where the category at the right consists of objects as above; and morphisms are defined as alluded to above. The assertion of the exercise above is that this functor is essentially surjective; one may also check it is fully faithful, and therefore an equivalence of categories.

1.6 Affine schemes

We have seen above that the structural sheaf $\mathcal{O}$ on $\operatorname{Spec} A$ recovers the ring A (and all its localizations). We will now isolate a key condition on morphisms the spectra of these rings, in such a way that we precisely recover ring homomorphisms. To do so, we need another concept from sheaf theory.

Definition 1.6.1. If F is a presheaf on a topological space X, and $x \in X$, the *stalk* of F is defined as

$$F_x := \operatorname{colim}_{x \in U} F(U).$$

Here the colimit runs over all open subsets U containing x, and for a smaller open neighborhood $U \supset V$, the transition maps are the restriction maps $F(U) \to F(V)$ (which are part of the datum of a presheaf).

Remark 1.6.2.

- More concretely, one can say that

$$F_x = \bigsqcup_{U \ni x} F(U)/\sim,$$

where $\sim$ is the equivalence relation generated by the relation that identifies $f \in F(U)$ with $g \in F(V)$ iff there is an open neighborhood $W \subset U \cap V$, $x \in W$ such that

$$f|_W = g|_W.$$

In prose: the stalk consists of sections of small open neighborhoods of x, where we identify two sections iff they agree on a possibly smaller neighborhood of x.

- It is also worth noting that if F is a presheaf (or sheaf) of rings, then F_x is a ring, too.
- A conceptual reason for this definition of the stalk is given by Exercise 1.6.27.

Example 1.6.3. If M is an A-module, we compute the stalks of the sheaf $\widetilde{M}$ (which includes as a special case the structural sheaf $\widetilde{A} = \mathcal{O}_{\operatorname{Spec} A}$) at a point $\mathfrak{p} \in \operatorname{Spec} A$. Since any open subset is a union of $D(f)$, we have

$$(\widetilde{M})_{\mathfrak{p}} = \operatorname{colim}_{\mathfrak{p} \in D(f)} \widetilde{M}(D(f)) = \operatorname{colim}_{f, f \notin \mathfrak{p}} M[f^{-1}] = M_{\mathfrak{p}} \ (:= M[(A \backslash \mathfrak{p})^{-1}]).$$

To see the right hand identification, note that $D(f) \subset D(g)$ holds iff g is a unit in $A[f^{-1}]$ (equivalently, the natural ring homomorphism $A \to A[f^{-1}]$ factors through $A[g^{-1}]$; Lemma 1.1.10(3)), the transition maps are the natural maps $M[g^{-1}] \to M[f^{-1}]$ in this event. In particular, we have

$$\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} = A_{\mathfrak{p}}.$$

Definition 1.6.4. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a *sheaf of rings* $\mathcal{O}_X$ on X. Here, sheaf "of rings" means a sheaf such that for each $U \subset X$, $\mathcal{O}_X(U)$ is a (commutative) ring, and the restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ are ring homomorphisms.

A *locally ringed space* is a ringed space such that the stalk $\mathcal{O}_{X,x}$ of the structural sheaf is a *local* ring, for each $x \in X$.

By Example 1.6.3, $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a locally ringed space. Exercise 1.6.25 offers an approach to define this notion without using stalks.

Example 1.6.5. Locally ringed spaces are an extremely broad notion. For example, if X is a topological space, then we may consider, for $U \subset X$ open:

$$\mathcal{O}_X(U) := \{f : U \to \mathbf{R} \text{ continuous}\}.$$

This clearly defines a sheaf of rings, and $(X, \mathcal{O}_X)$ is a locally ringed space. One checks this either directly from the definition (the maximal ideal in $\mathcal{O}_{X,x}$ consists of functions $f : V \to \mathbf{R}$ such that $f(x) = 0$). Alternatively, using Exercise 1.6.25: if $f : U \to \mathbf{R}$ is continuous, then $V := f^{-1}(\mathbf{R}\backslash\{0\})$ and $W := f^{-1}(\mathbf{R}\backslash\{1\})$ are open subsets and $U = V \cup W$.

Definition 1.6.6. Fix a continuous map of topological spaces $f : X \to Y$. The *direct image functor*

$$f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$$

is the functor given by precomposition with f^{-1}, i.e., for a sheaf F on X, f_*F is the sheaf defined by

$$f_*F(V) := F(f^{-1}(V)) \text{ for } V \subset Y \text{ open.}$$

(One immediately checks that f_*F, defined in this way, is indeed a sheaf, and that this construction of f_*F is functorial in F.)

Definition 1.6.7. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a map of sheaves of rings on Y

$$f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X.$$

Thus, by the definition of f_*, this means that for any open subset $V \subset Y$, there is a ring homomorphism

$$\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)),$$

which is required to be compatible with the restriction maps of $\mathcal{O}_Y$ and $\mathcal{O}_X$.

Example 1.6.8. The idea of the map $f^\sharp$ is that it takes a function that is defined on (an open subset V of) Y and in some sense composes that function with f in order to produce a function on (the open subset $f^{-1}(V)$ of) X.

To give more content to this idea, let $f : X \to Y$ be a continuous map between topological spaces, and consider the ringed spaces given by continuous functions (Example 1.6.5). Then there is a map (of sheaves on Y)

$$\mathcal{O}_Y \to f_*\mathcal{O}_X$$

whose evaluation at an open $V \subset Y$ is

$$\mathcal{O}_Y(V) \ni (g : V \to \mathbf{R}) \mapsto (g \circ f : f^{-1}(V) \to \mathbf{R}) \in \mathcal{O}_X(f^{-1}(V)) = (f_*\mathcal{O}_X)(V).$$

For a morphism of ringed spaces f as above, and any $x \in X$, and $y := f(x) \in Y$, we in particular have a map

$$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x},$$

Using the description of Remark 1.6.2, it takes a function $f \in \mathcal{O}_Y(V)$ for some neighborhood $V \ni y$, and takes its image under $f^\sharp$, which is a function in $\mathcal{O}_X(f^{-1}(V))$. (Note that $f^{-1}(V)$ is an open neighborhood of x.) If f is agrees with some other function $g \in \mathcal{O}_Y(U)$ on a possibly smaller neighborhood $W \subset U \cap V$, then their images under $f^\sharp$ will agree on $f^{-1}(W)$.

Definition 1.6.9. A *morphism of locally ringed spaces* is a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that the map

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \tag{1.6.10}$$

is a *local map* (Definition and Lemma 1.2.3) between the stalks (which are local rings by Definition 1.6.4). The category of locally ringed spaces with these morphisms is denoted by LocRingedSpace.

The motivation behind requiring the map (1.6.10) to be a local map is this: if we think of $\mathcal{O}_{Y,f(x)}$ to be of germs of functions, then the maximal ideal $\mathfrak{m}_{f(x)}$ corresponds to (germs of) functions that vanish at $f(x)$. The map $f^\sharp$ should send those (germs of) functions to ones that vanish at x.

Example 1.6.11. Given a ring homomorphism $f : A \to B$, we define a map

$$\varphi : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$$

on the level of the underlying spaces as in Lemma 1.1.3, and given on the level of sheaves by

$$\mathcal{O}_{\operatorname{Spec} A} \to \varphi_* \mathcal{O}_{\operatorname{Spec} B}$$

by using Lemma 1.5.2, which requires us to specify the map only on basic open subsets, where we define it to be the natural map

$$\mathcal{O}_{\operatorname{Spec} A}(D(a)) = A[a^{-1}] \to (\varphi_* \mathcal{O}_{\operatorname{Spec} B})(D(a)) = \mathcal{O}_{\operatorname{Spec} B}(\varphi^{-1}(D(a))) = \mathcal{O}_{\operatorname{Spec} B}(D(f(a))) = B[(f(a))^{-1}]$$

(i.e., $\frac{x}{a^n} \mapsto \frac{f(x)}{f(a)^n}$). Passing to stalks at a prime ideal $\mathfrak{q} \subset B$ induces the map

$$A_{f^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}},$$

which is a local map (since f maps the maximal ideal of $A_{f^{-1}(\mathfrak{q})}$ to the maximal ideal of $B_{\mathfrak{q}}$).

Non-example 1.6.12. Let p be a prime number and consider the map

$$f : (\operatorname{Spec} \mathbf{Q}, \mathcal{O}_{\operatorname{Spec} \mathbf{Q}}) \to (\operatorname{Spec} \mathbf{Z}, \mathcal{O}_{\operatorname{Spec} \mathbf{Z}})$$

given on the level of the underlying spaces by $\operatorname{Spec} \mathbf{Q} \ni (0) \mapsto (p)$ and on the level of functions by

$$\mathcal{O}_{\mathbf{Z}} \to f_* \mathcal{O}_{\mathbf{Q}}$$

the map whose evaluations on the fundamental open subsets $D(n)$, for $n \in \mathbf{Z}$, is given by the obvious map

$$\mathbf{Z}[n^{-1}] \to \begin{cases} \mathbf{Q} & p \nmid n \\ 0 & p | n \end{cases}$$

Passing to stalks at the prime ideal $\mathfrak{p} = (p)$, i.e., taking the colimit over all n such that $p \nmid n$ gives

$$\mathbf{Z}_{(p)} \to \mathbf{Q}.$$

This is *not* a local map: since p is not mapped to the maximal ideal in $\mathbf{Q}$ (which is the 0-ideal).

Example 1.6.11 establishes a functor

$$\operatorname{Spec} : \mathrm{Rings}^{\mathrm{op}} \to \mathrm{LocRingedSpace},$$

which is on objects given by $A \mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ and on morphisms by the above. We will henceforth abbreviate $\operatorname{Spec} A := (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, i.e., unless otherwise mentioned we will always regard $\operatorname{Spec} A$ as a locally ringed space (as opposed to its underlying topological space).

Proposition 1.6.13. The functor Spec is fully faithful. That is, there is a bijection

$$\operatorname{Hom}_{\mathrm{Rings}}(B, A) \xrightarrow{\cong} \operatorname{Hom}_{\mathrm{LocRingedSpace}}(\operatorname{Spec} A, \operatorname{Spec} B).$$

This is the special case $X = \operatorname{Spec} A$ of the next statement:

Theorem 1.6.14. There is an adjunction

$$\Gamma : \mathrm{LocRingedSpace} \rightleftarrows \mathrm{Rings}^{\mathrm{op}} : \operatorname{Spec},$$

where the *global sections functor* Γ sends a locally ringed space X to the ring $\Gamma(X, \mathcal{O}_X) := \mathcal{O}_X(X)$ and a map $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to its evaluation on global sections, i.e., to $f^\sharp(Y) : \mathcal{O}_Y(Y) \to \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(X)$.

Proof. For a ring B and a locally ringed space $(X, \mathcal{O}_X)$ we have to establish a bijection of Hom-sets as follows:

$$\mathrm{Hom}_{\mathrm{LocRingedSpace}}(X, \operatorname{Spec} B) \to \mathrm{Hom}_{\mathrm{Rings}^{\mathrm{op}}}(\mathcal{O}_X(X), B) = \mathrm{Hom}_{\mathrm{Rings}}(B, \mathcal{O}_X(X)) \tag{1.6.15}$$

This may be paraphrased by saying that a map (from any locally ringed space) to an affine scheme is determined by its value on global sections. Let us write $Y := (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$.

(1) Our first step is to prove the map is injective. Pick two elements $(f, f^\sharp)$ and $(g, g^\sharp)$ in the left hand set, such that the induced map $B := \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is the same. Denote this map by ϕ.

(a) We prove that $f = g$ (as a map of the underlying sets). Pick $x \in X$ and consider the diagram

$$\begin{tikzcd} B = \mathcal{O}_Y(Y) \arrow[r, "\phi"] \arrow[d] & \mathcal{O}_X(X) \arrow[d] \\ B_{\mathfrak{q}} = \mathcal{O}_{Y,f(x)} \arrow[r, "f^\sharp_x"] & \mathcal{O}_{X,x}. \end{tikzcd} \tag{1.6.16}$$

Here $\mathfrak{q} \subset B$ corresponds to $f(x) \in Y$. (The diagram is commutative by the functoriality of $f^\sharp$.) Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$. Since $f^\sharp_x$ is a *local* map, its preimage in $B_{\mathfrak{q}}$ is the maximal ideal of $B_{\mathfrak{q}}$, and so its preimage in B is $\mathfrak{q}$. If we replace f by g in there, the map ϕ and the right vertical map don't change, which shows that $f(x) = \mathfrak{q} = g(x)$.

(b) We now prove that $f^\sharp = g^\sharp$. By Lemma 1.5.2, it is enough to check these two morphisms of sheaves $\mathcal{O}_Y \to f_* \mathcal{O}_X$ agree on the basic open subsets $U = D(b)$ for $b \in B$. Again, consider a similar commutative diagram as above:

$$\begin{tikzcd} B = \mathcal{O}_Y(Y) \arrow[r, "\phi"] \arrow[d, "\mathrm{res}"] & \mathcal{O}_X(X) \arrow[r, "\cong"] \arrow[d, "\mathrm{res}"] & (f_*\mathcal{O}_X)(Y) \arrow[d, "\mathrm{res}"] \\ B[b^{-1}] = \mathcal{O}_Y(U) \arrow[r, "f^\sharp(U) \overset{?}{=} g^\sharp(U)"] & \mathcal{O}_X(f^{-1}(U)) \arrow[r, "\cong"] & (f_*\mathcal{O}_X)(U). \end{tikzcd} \tag{1.6.17}$$

Note that at the right we have $f_*\mathcal{O}_X = g_*\mathcal{O}_X$ since we already know $f = g$. To see that the bottom left maps agree, observe that a ring homomorphism out of $B[b^{-1}]$ is uniquely determined by its composition with $B \to B[b^{-1}]$, i.e., the left hand restriction map. This confirms $f^\sharp = g^\sharp$ and therefore that the map in (1.6.15) is injective.

(2) We now prove that the map in (1.6.15) is surjective. Given a ring homomorphism $\phi : B \to \mathcal{O}_X(X)$, we need to construct a map $(f, f^\sharp) : (X, \mathcal{O}_X) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$ of locally ringed spaces (whose global sections give back ϕ).

(a) Taking our cue from the above part of the proof, we define $f : X \to Y$ like so. Pick $x \in X$, consider $B \xrightarrow{\phi} \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$. The preimage of $\mathfrak{m}_x$ in B is a prime ideal $\mathfrak{q}$. We define $f(x) := \mathfrak{q}$.

(b) We check that the map f so defined is continuous. For a function $t \in \mathcal{O}_X(X)$, define a subset

$$D(t) := \{x \in X \mid t \in \mathcal{O}_{X,x}^\times\} \subset X.$$

This is an open subset: if $x \in D(t)$, then t is invertible when restricted to an open neighborhood $U \ni x$, so that $U \subset D(t)$. Note that if X happens to be affine, this precisely agrees with the previous definition of basic open subsets. The continuity of f is now easy: it suffices to check that $f^{-1}(D(b))$ is open for any $b \in B$. Indeed,

$$f^{-1}(D(b)) = \{x \in X \mid \phi(b) \in \mathcal{O}_{X,x}^\times\} = D(\phi(b)).$$

(c) Given the continuous map f, we define a homomorphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$, which again is enough to do on $D(b)$, $b \in B$. Consider again the diagram (1.6.17) above. We are required to provide the bottom left horizontal map. Again using the universal property of the localization $B[b^{-1}]$, there is a unique map making the diagram commute iff $\phi(b)$ is an invertible element in $\mathcal{O}_X(D(\phi(b)))$. We have just checked that this is indeed the case.

(d) We finally check that the map $(f, f^\sharp)$ of ringed spaces constructed so far is a map of *locally* ringed spaces. By our construction, the diagram (1.6.16) commutes. We need to see the map $f^\sharp_x$ is a local map, i.e., $(f^\sharp_x)^{-1}(\mathfrak{m}_x) = \mathfrak{q}B_\mathfrak{q}$. This preimage is certainly *some* prime ideal in $B_\mathfrak{q}$, and the map $\operatorname{Spec} B_\mathfrak{q} \to \operatorname{Spec} B$ is injective. So the commutativity of the diagram (1.6.16) and the definition of $\mathfrak{q}$ above finishes the job. □

Definition 1.6.18. We let the category AffSch of *affine schemes* be the essential image of the functor Spec. In other words, it is the full subcategory of LocRingedSpace consisting of those locally ringed spaces that are isomorphic to $\operatorname{Spec} A$, for some ring A.

Example 1.6.11 then asserts an equivalence of categories

$$\operatorname{Spec} : \mathrm{Rings}^{\mathrm{op}} \overset{\cong}{\rightleftarrows} \mathrm{AffSch} : \mathcal{O}(?), \tag{1.6.19}$$

where the right hand functor sends an affine scheme X to the ring $\mathcal{O}_X(X)$.

Thus, affine schemes are “nothing but” commutative rings. The full power of Proposition 1.6.13 will become visible once we introduce (non-affine) schemes, which are locally ringed spaces that are glued together from affine schemes. For now, the effect of Proposition 1.6.13 is that it gives us a way to interpret statements about commutative rings in a more geometric fashion: it allows us to switch back and forth between rings (and homomorphism between them) and their spectra (regarded, crucially, as locally ringed spaces, and morphisms of locally ringed spaces).

Definition 1.6.20. For a ring A and $n \geqslant 0$, the *affine n-space* or just *affine space* over A is defined as

$$\mathbf{A}^n_A := \operatorname{Spec} A[t_1, \dots, t_n].$$

For $n = 1$, we speak of the *affine line*. We usually abbreviate $\mathbf{A}^n := \mathbf{A}^n_\mathbf{Z}$.

Recall that there is a bijection

$$\mathrm{Hom}_{\mathrm{Rings}}(\mathbf{Z}[t], A) \underset{f \mapsto f(t)}{\overset{\cong}{\to}} A.$$

Indeed, a ring homomorphism $\mathbf{Z}[t] \to A$ is uniquely determined by its value on t, and this element of A can be chosen freely. Reinterpreting this in light of Proposition 1.6.13, there is a bijection

$$\mathrm{Hom}_{\mathrm{AffSch}}(\operatorname{Spec} A, \mathbf{A}^1) = A.$$

Thus, functions (in the sense of algebraic geometry) on $\operatorname{Spec} A$ are just the elements of A. Similarly, n-tuples $(a_1, \dots, a_n)$ of elements of A are nothing but maps $\operatorname{Spec} A \to \mathbf{A}^n$. We will use this insight to phrase conditions about elements in A in geometric terms, such as Exercise 1.6.23 which gives a(n obvious) geometric reinterpretation of the condition of being an integral domain.

Definition 1.6.21. For a ring A, we define

$$\mathbf{G}_{\mathrm{m},A} := \operatorname{Spec} A[t^{\pm 1}],$$

and again write $\mathbf{G}_\mathrm{m} := \mathbf{G}_{\mathrm{m},\mathbf{Z}}$. For reasons explained in Exercise 1.6.33, this is referred to as the *multiplicative group*.

Note that

$$\mathbf{G}_{\mathrm{m}} = D(t) \subset \mathbf{A}^1 \tag{1.6.22}$$

is an open subscheme, namely the complement of $V(t)$, which we will refer to as the *origin* of $\mathbf{A}^1$. Similarly to the above, there are bijections

$$\mathrm{Hom}_{\mathrm{AffSch}}(\mathrm{Spec}\, A, \mathbf{G}_{\mathrm{m}}) = \mathrm{Hom}_{\mathrm{Rings}}(\mathbf{Z}[t^{\pm 1}], A) \underset{f \mapsto f(t)}{\overset{\cong}{\to}} A^{\times}.$$

Exercises

Exercise 1.6.23. Let $C = \mathrm{Spec}\, \mathbf{Z}[t,u]/tu$. Recall the two irreducible components $C_1, C_2 (\cong \mathbf{A}^1)$ of C (Exercise 1.1.27). Recall that a *domain* is a (commutative) ring A such that $ab = 0$ implies $a = 0$ or $b = 0$. Let A be a commutative ring such that $\mathrm{Spec}\, A$ is connected.

Prove that A is a domain iff for any horizontal arrow there is a diagonal arrow making the triangle commute (the right vertical map is the natural map induced by the inclusions $C_i \subset C$):

$$\begin{tikzcd} & C_1 \sqcup C_2 \arrow[d] \\ \mathrm{Spec}\, A \arrow[r] \arrow[ur, dotted] & C. \end{tikzcd}$$

Here $C_1 \sqcup C_2$ is the coproduct in the category AffSch; by Proposition 1.6.13 finite coproducts of affine schemes correspond to finite products of rings, i.e., we have $C_1 \sqcup C_2 = \mathrm{Spec}(\mathbf{Z}[u] \times \mathbf{Z}[t])$.

Remark 1.6.24. The assumption that $\mathrm{Spec}\, A$ be connected can be removed if one considers the category $\mathrm{AffSch} \cong \mathrm{Rings}^{\mathrm{op}} \xrightarrow{y} \mathrm{Fun}(\mathrm{Rings}, \mathrm{Set})$, where y is the Yoneda embedding that takes any ring R to the functor $y(R) : \mathrm{Rings} \to \mathrm{Set}$ given by $S \mapsto \mathrm{Hom}_{\mathrm{Rings}}(R, S)$ (but y does not respect coproducts). By general category theory, y is a fully faithful functor. Now, prove that A is a domain iff the lifting condition below is satisfied:

$$\begin{tikzcd} & y(C_1) \sqcup y(C_2) \arrow[d] \\ y(\mathrm{Spec}\, A) \arrow[r] \arrow[ur, dotted] & y(C) \end{tikzcd}$$

Exercise 1.6.25. Let $(X, \mathcal{O}_X)$ be a ringed space. Prove that the following are equivalent:

(1) All stalks $\mathcal{O}_{X,x}$ are local rings, i.e., it is a locally ringed space.

(2) The following two conditions are satisfied:

  (a) The only open $U \subset X$ such that $\mathcal{O}_X(U) = 0$ is $U = \varnothing$.

  (b) If $f \in \mathcal{O}_X(U)$ then $U = V \cup W$ with two open subsets $V, W \subset U$ (it is allowed that one of them is empty) such that $f|_V$ is invertible and $(1-f)|_W$ is invertible.

(Hint: it is convenient to use the characterization of local rings in Definition and Lemma 1.2.1(4).) $\blacksquare$

Exercise 1.6.26. Use Exercise 1.6.25 to give another proof of the fact that $(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$ is a locally ringed space.

Exercise 1.6.27. Let X be a topological space and $x \in X$. Consider the obvious map $i_x : \{\star\} \to X$ sending $\star$ to x. Prove that the functor $\mathrm{Shv}(X) \to \mathrm{Set}$, $F \mapsto F_x$ is left adjoint to the direct image functor (along the map i_*):

$$\mathrm{Set} = \mathrm{Shv}(\{*\}) \xrightarrow{(i_x)_*} \mathrm{Shv}(X).$$

Exercise 1.6.28. Let $\varphi : F \to G$ be a map of sheaves on a topological space X. Prove that φ is an isomorphism (i.e., $F(U) \to G(U)$ is an isomorphism for all open $U \subset X$) iff for all $x \in X$ the induced map on stalks, $F_x \to G_x$, is an isomorphism.

Exercise 1.6.29. Let $(X, \mathcal{O}_X)$ be a locally ringed space and $U \subset X$ an open subset of the underlying topological space X.

(1) Show that $(U, \mathcal{O}_X|_U)$ is naturally locally ringed space as well.

(2) Let $(Y, \mathcal{O}_Y)$ be another locally ringed space. Prove that a morphism $(f, f^\sharp) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ factors uniquely (as a morphism of locally ringed spaces!) over $(U, \mathcal{O}_X|_U)$ provided that $f(Y) \subset U$. (In more formulaic terms,

$$\mathrm{Hom}_{\mathrm{LocRingedSpace}}((Y, \mathcal{O}_Y), (U, \mathcal{O}_X|_U)) = \mathrm{Hom}_{\mathrm{LocRingedSpace}}((Y, \mathcal{O}_Y), (X, \mathcal{O}_X)) \times_{\mathrm{Hom}_{\mathrm{Top}}(Y,X)} \mathrm{Hom}_{\mathrm{Top}}(Y, U).)$$

Exercise 1.6.30. The following foundational statement, which is in a sense complementary to the characterization of maps *into* an open subspace offered by Exercise 1.6.29(2), is often referred to by saying that maps (of locally ringed spaces) *glue*.

Let $(X, \mathcal{O}_X)$ be a locally ringed space and $X = \bigcup_i U_i$ a cover by open subsets. Put $U_{ij} := U_i \cap U_j$ Consider the induced locally ringed spaces $(U_i, \mathcal{O}_{U_i})$ as in Exercise 1.6.29, and similarly for U_{ij}. Hereafter we write $X := (X, \mathcal{O}_X)$ etc. For any locally ringed space $(Y, \mathcal{O}_Y)$, establish a bijection

$$\mathrm{Hom}_{\mathrm{LocRingedSpace}}(X, Y) = \{(f_i) \in \mathrm{Hom}_{\mathrm{LocRingedSpace}}(U_i, Y) \mid f_i|_{U_{ij}} = f_j|_{U_{ij}}\}.$$

Note that this is saying that X is a colimit (in the category of locally ringed spaces) of the diagram

$$\bigsqcup_{i,j} U_{ij} \rightrightarrows \bigsqcup_i U_i,$$

where the two maps are the inclusion of U_{ij} into U_i and U_j, respectively.

Exercise 1.6.31. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces. Prove that the following are equivalent:

(1) For all $x \in X$, the maps $\varphi : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induced by f are local ring homomorphisms in the sense that $\varphi^{-1}(\mathcal{O}_{X,x}^\times) = \mathcal{O}_{Y,f(x)}^\times$.

(2) f is a map of locally ringed space in the sense of Definition 1.6.9.

Exercise 1.6.32. Let $X = \mathbf{A}^2 = \operatorname{Spec} \mathbf{Z}[t_1, t_2]$. Let $U \subset X$ be the *punctured plane*, i.e., the complement of the origin, which is the closed point (equivalently, maximal ideal) $(t_1, t_2) \in \mathbf{A}^2$. Yet another way to say this is $U = D(t_1) \cup D(t_2)$. Show that the restriction map

$$(\mathbf{Z}[t_1, t_2] =)\ \mathcal{O}_X(X) \stackrel{\mathrm{res}_X^U}{\to} \mathcal{O}_X(U)$$

is an *isomorphism*.

In other words, every regular function on U can be (uniquely) extended to one on X. This situation is identical to *Hartog's theorem* in complex analysis which states that a holomorphic function on $\mathbf{C}^2 \backslash \{(0,0)\}$ can be extended to a holomorphic function on $\mathbf{C}^2$; in contrast to what happens on $\mathbf{C} \backslash \{0\}$.

Hint: let $U_1 = D(t_1) = \operatorname{Spec} \mathbf{Z}[t_1^{\pm 1}, t_2]$ and define U_2 similarly. Inspect the exact sequence (1.5.1).

Exercise 1.6.33. (1) Show that for any affine scheme X, there are natural bijections

$$\mathrm{Hom}_{\mathrm{AffSch}}(X, \mathbf{A}^1) \stackrel{\cong}{\to} \Gamma(X, \mathcal{O}_X).$$

$$\mathrm{Hom}_{\mathrm{AffSch}}(X, \mathbf{G}_{\mathrm{m}}) \stackrel{\cong}{\to} \Gamma(X, \mathcal{O}_X^\times).$$

(2) Deduce from the Yoneda lemma that $\mathbf{A}^1$ has the structure of a ring (and $\mathbf{G}_\mathrm{m}$ the structure of an abelian group object) in AffSch (i.e., there is a "sum" and a "multiplication" map $\mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$, and a "negative" map $\mathbf{A}^1 \to \mathbf{A}^1$ satisfying the usual ring axioms). One refers to this by saying that $\mathbf{A}^1$ (which is in this context also denoted by $\mathbf{G}_\mathrm{a}$ for *additive group*) and $\mathbf{G}_\mathrm{m}$ are (abelian) affine *group schemes*.

(3) Describe concretely the ring homomorphisms

$$\mathbf{Z}[t] \to \mathbf{Z}[t] \otimes \mathbf{Z}[t], \text{ (resp. } \mathbf{Z}[t^\pm] \to \mathbf{Z}[t^\pm] \otimes \mathbf{Z}[t^\pm]),$$

that correspond to the addition on $\mathbf{A}^1$ (respectively the multiplication on $\mathbf{G}_\mathrm{m}$)?

(4) Alternatively, construct the group structure on $\mathbf{G}_\mathrm{m}, \mathbf{A}^1 \in \mathrm{AffSch}$ by proving an adjunction

$$\mathbf{Z}[-] : \mathrm{AbMon} \rightleftarrows \mathrm{Rings} : (-)^\times,$$

where the left adjoint sends an abelian monoid to the group ring, and the right adjoint sends a (commutative) ring to its group of units. Observe that $\mathbf{Z}[t] = \mathbf{Z}[\mathbf{N}]$, $\mathbf{Z}[t^{\pm 1}] = \mathbf{Z}[\mathbf{Z}]$.

1.7 Integrality and valuation rings

Definition 1.7.1. We say that an A-module M is *finite* (or A-finite if we want to emphasize the ring A) if it is generated by finitely many elements $m_1, \dots, m_n \in M$. (I.e., there is a surjection of A-modules, $\bigoplus_{i=1}^n A \to M$.)

We say that an A-algebra B is *finite* if it is finite as an A-module. We also say that $A \to B$ is a finite map in this case.

Remark 1.7.2. Note that this is much stronger than requiring B to be finitely generated as an A-algebra: the polynomial ring $A[t]$ is a finitely generated A-algebra, but not finitely generated as an A-module; see Exercise 1.7.28 for a precise assertion pinpointing the difference between the two.

Definition and Lemma 1.7.3. Let B be a ring, $A \subset B$ a subring and $b \in B$ an element. The following conditions are equivalent; if they are satisfied we say that b is *integral* over A.

(1) $A[b]$ is contained in an subalgebra $B' \subset B$ that is finite over A. Here and below, $A[b]$ denotes the A-subalgebra of B generated by b, i.e., the image of the map $A[t] \to B$, $t \mapsto b$.

(2) b is the zero of a monic polynomial with coefficients in A, i.e., there are $a_0, \dots, a_{n-1} \in A$ such that

$$b^n + a_{n-1}b^{n-1} + \dots + a_0 = 0.$$

(3) $A[b]$ is a finite A-module.

Proof. (1) ⇒ (2): let B' be as stated, with $y_1 = 1, y_2, \dots, y_n$ being finitely many generators of B' (as an A-module). For all i, we have $B' \ni by_i = \sum_j a_{ij} y_j$ for some $a_{ij} \in A$, so that

$$y_i \det(b\mathrm{id} - (a_{ij})) = 0.$$

For $i = 1$ and $y_1 = 1$ we get the requested monic equation for b.

(2) ⇒ (3) If $b^n + a_{n-1}b^{n-1} + \dots + a_0 = 0$, then $A[b]$ is generated by $1, b, \dots, b^{n-1}$.

(3) ⇒ (1) is trivial. □

Definition 1.7.4. An *integral extension* is an injective ring homomorphism $A \subset B$ such that any $b \in B$ is integral over A.

An *integral ring homomorphism* (or *integral map*) is a ring homomorphism $f : A \to B$ such that the induced map $f(A) \cong A/\ker f \subset B$ is an integral extension.

A cheap example of a non-injective integral map is $A \to B := A/I$. Geometrically this corresponds to a closed subscheme. We will mostly be concerned with integral extensions in the sequel.

Corollary 1.7.5. Let $A \subset B \subset C$ be given. If $A \subset B$ and $B \subset C$ are integral extensions, then so is the composite $A \subset C$.

Proof. Let $c \in C$. There is a monic polynomial $p(t) \in B[t]$ with $p(c) = 0$. Its coefficients generate (since $A \subset B$ is integral) a finite A-submodule of C, denoted B'. Then $B'[c]$ is a finite A-module, so c is integral over A (Definition and Lemma 1.7.3(1)). $\square$

Corollary 1.7.6. If $A \subset B$ as above, then the subset

$$\widetilde{A} := \{b \in B \mid b \text{ is integral over } A\}$$

forms a subring. It is called the *integral closure* of A in B. We say that A is *integrally closed* in B if $\widetilde{A} = A$.

Proof. If $b, b' \in \widetilde{A}$, then $A[b, b']$ is finite over A (by applying Definition and Lemma 1.7.3 twice). Thus $bb', b + b'$ are contained in an A-finite subalgebra, so they are integral, again by Definition and Lemma 1.7.3. $\square$

Example 1.7.7. The integral closure is of paramount importance in number theory: given a finite extension $K/\mathbf{Q}$, one studies there the ring $\mathcal{O}_K$, the integral closure of $\mathbf{Z}$ in K.

Lemma 1.7.8. For a subring $A \subset B$ the following are equivalent:
(1) It is an integral extension (i.e., any $b \in B$ is integral over A or $\widetilde{A} = B$).

(2) B is a filtered colimit of A-subalgebras that are finite A-modules.

Proof. For *any* $A \subset B$, we have

$$B = \operatorname{colim}_{S \subset B \text{ finite}} A[S],$$

where the (filtered) colimit runs over all the finite subsets of B, and $A[S]$ denotes the A-subalgebra generated by S. Moreover, any $b \in B$ lies in the subalgebra $A[b]$. Given these prerequisites, the statement is now an immediate consequence of Definition and Lemma 1.7.3: if any b is integral, i.e., $A[b]$ is finite, then by induction $A[S]$ is finite for any finite subset $S \subset B$. Conversely, if $B = \operatorname{colim} B_i$ for some A-finite subalgebras, then any $b \in B$ (and therefore $A[b]$) lies in some B_i. That is, b is integral over A. $\square$

Definition and Lemma 1.7.9. Let A be a ring, with $A \neq \{0\}$. For an ideal $I \subset A$, the following conditions are equivalent:
(1) for all $f \in I$, $1 + f \in A^\times$,

(2) for all finite A-modules M we have $IM = M$ (equivalently $M/IM = 0$) if and only if $M = 0$,

(3) I is contained in every maximal ideal.
There is a largest ideal satisfying these equivalent conditions, it is called the *radical* or *Jacobson radical* (not to be confused with the nilradical or the radical of an ideal in A, cf. (1.1.4)), and denoted by $\operatorname{rad} A$.

Proof. (1) $\Rightarrow$ (2): Assume $M \neq 0$ and pick a minimal system of generators $m_1, \dots, m_n$. By assumption $m_1 = \sum a_i m_i$ for $a_i \in I$, so $(1 - a_1)m_1$ lies in the span of $m_2, \dots, m_n$. But $1 - a_1 \in A^\times$, hence the system was not minimal, giving a contradiction.

(2) $\Rightarrow$ (3): If $\mathfrak{m} \subset A$ is a maximal ideal, the A-module $M := A/\mathfrak{m}$ is a field and therefore *simple*, i.e., any submodule is either 0 or equal to M. Since $A \neq 0$ we have $M \neq 0$ and therefore, by (2), $IM \subsetneq M$, so $IM = 0$, i.e., $I \subset \mathfrak{m}$.

(3) $\Rightarrow$ (1): If $f \in I$ was such that $1 + f \notin A^\times$, then the principal ideal $(1 + f)$ would be contained in some maximal ideal $\mathfrak{m}$, so $f \notin \mathfrak{m}$, contradicting (3). $\square$

Note that for a local ring A, we have $\operatorname{rad} A = \mathfrak{m}$ (the maximal ideal). The statement (2) is then a bread-and-butter result in commutative algebra, known as the *Nakayama lemma*:

Lemma 1.7.10. For a local ring A, a finite A-module M is zero iff $M/\mathfrak{m}M (= M \otimes_A k) = 0$ (where $k = A/\mathfrak{m}$ is the residue field of A).

The next two statements will be used in the next section on valuation rings.

Lemma 1.7.11. Let $A \subset B$ be an integral extension. Then $\operatorname{rad} A \subset \operatorname{rad} B$.

Proof. By Definition and Lemma 1.7.9(1), we have to show that for (finitely many) $a_i \in \operatorname{rad} A$ and $b_i \in B$ we have $1 + \sum_i a_i b_i \in B^\times$. Let $B' := A[b_1, \dots, b_n]$. By integrality, it is a finite A-module, and it is enough to show $1 + \sum a_i b_i \in B'^\times$. We can therefore replace B by B' and suppose B is a finite A-module.

Let $I := (\operatorname{rad} A) \cdot B$. In order to show $I \subset \operatorname{rad} B$ we use Definition and Lemma 1.7.9(2): it is enough to show that for a finite B-module M, $M = IM$ implies $M = 0$. But $IM = (\operatorname{rad} A)M$, where here at the right M is regarded as an A-module. M is (since B is finite over A) a finite A-module, so again using the characterization of the radical (this time of A), we have $M = 0$, as desired. $\square$

Lemma 1.7.12. Let again $A \subset B$ and $b \in B$. If $1 \in (\operatorname{rad} A)A[b]$ (the ideal in $A[b]$ generated by $\operatorname{rad} A$), then b is invertible in $A[b]$, and this inverse b^{-1} is integral over A.

Proof. Let $1 = a + rb$ with $a \in \operatorname{rad}(A)$ and $r \in A[b]$. Since $1 - a \in A^\times$, say $a'(1-a) = 1$ we have $1 = a'rb$, so b is invertible in $A[b]$ with $b^{-1} = a'r = \sum_{i=0}^n a_i b^i$, so that $b^{-(n+1)} = \sum a_i b^{-(n-i)}$, so b^{-1} is indeed integral over A. $\square$

1.7.1 Valuation rings

Recall from Definition and Lemma 1.2.3 the definition of a local map between local rings. In the sequel we will be considering injective homomorphisms $A \subset B$ (between local rings). Such an inclusion is local if

(1) $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ or, equivalently,

(2) $\mathfrak{m}_A \subset \mathfrak{m}_B$ or, yet equivalently,

(3) $1 \notin \mathfrak{m}_A B$.

Proposition 1.7.13. Let $A \subset K$ be a ring contained in a field K. The following statements are equivalent; if they hold we call A a *valuation ring*.

(1) For any $x \in K$ we have $x \in A$ or $x^{-1} \in A$ (or both).

(2) A is a local ring and is maximal among the local subrings of K in the sense that if $A \subset B \subset K$, with B a local ring and the inclusion $A \subset B$ a local homomorphism, then $A = B$.

(3) A is local and for any intermediate ring $A \subsetneq C \subset K$, $\mathfrak{m}_A C = C$ (i.e., 1 lies in the ideal (in C) generated by $\mathfrak{m}_A$).

Proof. We first prove (3) $\Leftrightarrow$ (2). The implication $\Rightarrow$ is trivial in view of the above discussion of injective local homomorphism. Conversely, we prove (2) $\Rightarrow$ (3): If $\mathfrak{m}_A C \subsetneq C$, then there would be a prime ideal $\mathfrak{p}$ in C with $\mathfrak{m}_A C \subset \mathfrak{p}$. We would then have a proper inclusion $A \subsetneq C_\mathfrak{p}$ into a larger local ring.

(2) $\Rightarrow$ (1): let $x \in K \backslash A$. The subring $A[x] \subset K$ generated by x is larger than A. By (3), $1 \in \mathfrak{m}_A A[x] = (\operatorname{rad} A)A[x]$. By Lemma 1.7.12, x^{-1} is integral over A. By Lemma 1.7.11 (applied to $A \subset A[x^{-1}]$), $1 \notin \mathfrak{m}_A A[x^{-1}]$, so again using (2) $\Leftrightarrow$ (3), we have $A[x^{-1}] = A$, so $x^{-1} \in A$.

(1) $\Rightarrow$ (2): to see that A is local, let $a \in A$. If $x := (1-a)/a = a^{-1} - 1 \in A$ then a is invertible. Otherwise $x^{-1} = \frac{a}{1-a} \in A$ so, by symmetry, $1 - a$ is invertible.

If $A \subsetneq B \subset K$, we pick $b \in B \backslash A$. Then $b^{-1} \in A$ and therefore $b^{-1} \in \mathfrak{m}_A$, then $1 = bb^{-1} = \mathfrak{m}_A \cdot B$. $\square$

Note that in the event that (1) holds, $K = Q(A)$ (the quotient field of A). The following statement yields a rich supply of valuation rings.

Proposition 1.7.14. Let $A \subset K$ be a local subring of a field K. There is a valuation ring V such that

$$A \subset V \subset K$$

and the inclusion $A \subset V$ is local (one also says that V *dominates* A).

Proof. Consider the set of such factorizations

$$A \subset B \subset K$$

with B being local and $A \subset B$ a local inclusion.

If $(B_i)_{i \in I}$ is a totally ordered family of such intermediate local subrings of K, then $B := \bigcup_{i \in I} B_i$ is again a local subring of K (any element $b \in B$ lies in one of the B_i's, so $b \in B_i^\times$ or $1 - f \in B_i^\times$, and hence similarly in B). Also, the inclusion $A \subset B$ is local: if $a \in A$ becomes a unit in B, then it is a unit in one of the B_i, so $a \in A^\times$.

Zorn's lemma implies the existence of a maximal element among such factorizations; by Proposition 1.7.13, this is nothing but a valuation ring. □

This construction implies the following geometric key property of valuation rings. Recall that valuation rings V are local domains, so that (0) is a prime ideal; also there is unique maximal ideal $\mathfrak{m}_V$. These ideals correspond to the generic point, denoted η, and the unique closed point, denoted s, respectively.

Corollary 1.7.15. [Stacks, Tag 01J8] Let A be a ring and $\mathfrak{p} \subset \mathfrak{q}$ a containment of prime ideals (i.e., $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ a specialization in $\operatorname{Spec} A$). Then there is a valuation ring V and a map as displayed:

$$\begin{tikzcd}
\eta \arrow[r, rightsquigarrow] \arrow[d, mapsto] & s \arrow[d, mapsto] & \in \operatorname{Spec} V \arrow[d] \\
\mathfrak{p} \arrow[r, rightsquigarrow] & \mathfrak{q} & \in \operatorname{Spec} A.
\end{tikzcd}$$

In addition, if $k(\mathfrak{p}) \subset k'$ is a field extension, we may find V in such a way that $k(\eta) = k'$ (more precisely, the field extension $k(\mathfrak{p}) \subset k(\eta)$ of the residue fields induced by our map is isomorphic to the given extension).

Proof. The given data yields a map

$$A_{\mathfrak{q}} \to A_{\mathfrak{p}} \to k(\mathfrak{p}) \subset k'.$$

According to Proposition 1.7.14, we can obtain a commutative diagram

$$\begin{tikzcd}
A_{\mathfrak{q}} \arrow[rr, hook] \arrow[dr, hook, "f'"'] & & k' \\
& V \arrow[ur, hook, "f''"'] &
\end{tikzcd}$$

where

- f' is local: by the discussion at the beginning of §1.7.1, this means $f'^{-1}(\mathfrak{m}_V) = \mathfrak{m}_{A_{\mathfrak{q}}}$, i.e., that $\mathfrak{m}_V \mapsto \mathfrak{q}$.
- f'' is injective: this is equivalent to $f''^{-1}(0) = (0)$. Thus, $\operatorname{Spec} Q(A) \to \operatorname{Spec} V$ maps the unique point to the generic point of $\operatorname{Spec} V$, which is therefore mapped to the generic point of $\operatorname{Spec} A_{\mathfrak{q}}$, i.e., to $\mathfrak{q}$. □

Let us inspect some basic properties of valuation rings.

Lemma 1.7.16. The following are equivalent:
(1) A is a valuation ring.
(2) A is a local domain, and every finitely generated ideal is principal (i.e., generated by a single element).

Proof. (1) $\Rightarrow$ (2): To show A is local, we use the characterization in Definition and Lemma 1.2.1(4): let $f \in A$. Then $f|1-f$ or $1-f|f$. By symmetry we may assume $f|1-f$, i.e., there is some $a \in A$ with $af = 1-f$. Thus $(1+a)f = 1$, so f is a unit.

If $I = (f_1, \dots, f_n)$ we have $f_2|f_1$ (in which case $I = (f_2, \dots, f_n)$) or $f_1|f_2$ (in which case $I = (f_1, f_3, \dots, f_n)$ etc.

(2) $\Rightarrow$ (1): Let $a, b \in A$. The ideal (a, b) is principal, say, $(a, b) = (x)$, i.e., $a = \alpha x$, $b = \beta x$ and $x = ea + fb$ for $\alpha, \beta, e, f \in A$. Then $e\alpha x + f\beta x = x$, i.e., $e\alpha + f\beta = 1$ since A is a domain. Since A is local we have, say, $e\alpha \in A^\times$, so $\alpha \in A^\times$ and $b = \beta\alpha^{-1}a$. □

Corollary 1.7.17. Let A be a Noetherian integral domain. Then A is a valuation ring iff it is a local principal ideal domain (abbreviation: PID). If it is not a field, such a ring is called a *discrete valuation rings* (abbreviation: DVR).

For example, for a field k, the localizations $k[t]_{\mathfrak{p}}$ are DVRs. The ring $\mathbf{Z}_{(p)}$ above is also a DVR, as is $\mathbf{Z}_p$, the ring of *p-adic numbers*, for example. For a DVR A, we have $\operatorname{Spec} A = \{\eta, s\}$, where $\eta = (0)$ is the generic point, and s is any non-zero prime ideal, which is the unique maximal ideal (i.e., closed point in $\operatorname{Spec} A$). Another example (not used in the sequel) of a discrete valuation ring is the stalk of the sheaf of holomorphic functions on a Riemann surface X: given $x \in X$ and a meromorphic function $f(z)$ defined locally around x, a) f is holomorphic a given point x or b) f has a pole of order n, so that f is non-zero in a neighborhood of x, so that $\frac{1}{f}$ is holomorphic in a neighborhood of x.

Another perspective on this remark is that a valuation ring is either a principal ideal domain or non-Noetherian! This also explains to an extent why the construction of valuation rings in Proposition 1.7.14 is somewhat indirect.

Lemma 1.7.18. Any valuation ring A is *integrally closed* in its field of fractions $Q(A)$, i.e., any $c \in Q(A)$ that satisfies a monic polynomial equation

$$c^n + a_{n-1}c^{n-1} + \dots + a_0 = 0 (\in Q(A)),$$

for appropriate $a_i \in A$, already lies in A.

Proof. Since A is a valuation ring, we have $c \in A$ or $c^{-1} \in A$, cf. Proposition 1.7.13(1). In the latter case, we divide the given equation by c^{n-1} so that

$$-c = a_{n-1} + a_{n-2}c^{-1} + \dots + a_0 c^{-n+1}$$

lies in A. □

For example, any *unique factorization domain* (abbreviation: UFD) is integrally closed (Exercise 1.7.25).

The following characterization of universally closed morphisms is a first stepping stone towards the notion of *proper* morphisms between (not necessarily affine) schemes that we will study later on.

Lemma 1.7.19. For a map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ (induced by a ring homomorphism $f : A \to B$), consider $T := \operatorname{im} \varphi$. If T is stable under specialization, then T is closed.

Proof. Let $I := \ker f$. We may replace A by A/I (since $\operatorname{Spec} A/I$ is closed in $\operatorname{Spec} A$), so we may assume f is injective. We claim that φ is surjective then.

In general, for f injective, $\operatorname{im} \varphi$ contains the *minimal* primes of A (for such a prime $\mathfrak{p}$, $\operatorname{Spec} A_{\mathfrak{p}}$ is a singleton. Its preimage, $\varphi^{-1}(\mathfrak{p}) = \operatorname{Spec} B_{\mathfrak{p}}$ (Lemma 1.2.6), is non-zero, since it arises from the injection $A \subset B$ by localization, which is an exact functor).

Now if T is stable under specialization, it contains with all the minimal primes all prime ideals.□

Theorem 1.7.20. Consider a ring homomorphism $f : A \to B$ and denote by $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ the induced map on spectra. The following are equivalent:

(1) f is an integral map (not necessarily injective, i.e., $f(A) \subset B$ is an integral extension),

(2) f satisfies the lifting property as shown, i.e., for each valuation ring V (and its quotient field $Q(V)$) and each commutative outer square there is a diagonal map such that the two triangles commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & V \arrow[d, hook] \arrow[dl, dotted, "\exists"] \\ B \arrow[r] & Q(V) \end{tikzcd}$$

Equivalently, in the category of affine schemes, φ satisfies the lifting property as shown:

$$\begin{tikzcd} \operatorname{Spec} Q(V) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d, "\varphi"] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dotted, "\exists"] & \operatorname{Spec} A \end{tikzcd}$$

(3) $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is universally closed, i.e., for any algebra map $A \to A'$, the induced map $\operatorname{Spec}(B \otimes_A A') \to \operatorname{Spec} A'$ is closed.

(4) $\mathbf{A}^1_B = \operatorname{Spec} B[t] \to \mathbf{A}^1_A = \operatorname{Spec} A[t]$ is closed,

Proof. (1) ⇒ (2): Contemplate the following diagram:

$$\begin{tikzcd} Q(V) & B \arrow[l, "\varphi"] \arrow[dl, dotted, "\exists"] \\ V \arrow[u, hook] & A \arrow[l, "\psi"] \arrow[u, "f"] \end{tikzcd}$$

Since any $b \in B$ satisfies a monic equation (with coefficients in (the image of) A), $c := \varphi(b)$ satisfies a monic equation (with coefficients in V). Since V is integrally closed in $Q(V)$ (Lemma 1.7.18), $c \in V$, so the diagonal arrow exists.

(2) ⇒ (3): since morphisms satisfying the lifting property are stable under pullbacks (of affine schemes), it is enough to show that a map $A \to B$ satisfying (2) is closed (on spectra). Let $Z \subset \operatorname{Spec} B$ be closed. We have to check that $\varphi(Z)$ is closed. By Lemma 1.7.19, it is enough to show $f(Z)$ is stable under specialization. Let $z \in Z$ and $\varphi(z) = \mathfrak{p} \rightsquigarrow \mathfrak{q}$ be a specialization. By Corollary 1.7.15 there is a valuation ring V with fraction field $Q(V) = k(z)$ and the depicted bottom horizontal map satisfying $\eta_V \mapsto \mathfrak{p}$, $s_V \mapsto \mathfrak{q}$

$$\begin{tikzcd} \operatorname{Spec} k(z) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d, "\varphi"] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dotted, "\psi"] & \operatorname{Spec} A \end{tikzcd}$$

By (2), there is the diagonal dotted map ψ, so the closed point $\eta \rightsquigarrow s$ in $\operatorname{Spec} V$ lifts to a specialization $z = \psi(\eta) \rightsquigarrow \psi(s)$, which in turn maps to the given specialization $\mathfrak{p} \rightsquigarrow \mathfrak{q}$. Therefore, $\mathfrak{q} \in \operatorname{im} \varphi$.

(3) $\Rightarrow$ (4) is trivial: take $A' = A[t]$.

(4) $\Rightarrow$ (1) (following [Oli83, Theorem 3.2]) pick $b \in B$ and consider the map $B[t] \to B[b^{-1}]$ sending t to b^{-1}. Let C be the image of the composite $A[t] \to B[t] \twoheadrightarrow B[b^{-1}]$. We have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B[b^{-1}] := \operatorname{Spec} B[t]/bt-1 \arrow[r] \arrow[d] & \mathbf{A}^1_B \arrow[d] \\ \operatorname{Spec} C \arrow[r] & \mathbf{A}^1_A \end{tikzcd}$$

where the two horizontal maps are closed embeddings (since $A[t] \twoheadrightarrow C$). By (4), the right hand and therefore also the left hand vertical map is closed. In addition, $C \to B[b^{-1}]$ is injective. By Lemma 1.7.22, the map is therefore conservative, so that t, which is invertible in $B[b^{-1}]$, is already invertible in the subring $C \subset B[b^{-1}]$. That is, its inverse $t^{-1} = b$ is of the form $b = \sum_{i=0}^n a_i (b^{-1})^i$ for some $a_i \in A$. Multiplying with b^n shows that b satisfies a monic polynomial equation with coefficients in A. $\square$

Remark 1.7.21. The proof of (1) $\Rightarrow$ (2) actually shows that the (1) implies the existence *and unicity* of the diagonal map. We have refrained from stating this above, since for maps between general schemes the condition of being universally closed will turn out to be equivalent to (only) the existence of the lift.

Lemma 1.7.22. Let $f : A \to B$ be an injective ring homomorphism such that $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is closed. Then f is conservative, i.e., it satisfies the lifting property as shown (equivalently, any $a \in A$ such that $f(a) \in B^\times$ already satisfies $a \in A^\times$:

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d, "\varphi"] & \mathbf{G}_{\mathrm{m}} \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dotted, "\exists!"] & \mathbf{A}^1. \end{tikzcd}$$

Proof. If f is injective then φ has dense image (Lemma 1.1.12); thus by closedness φ is surjective. Thus the lifting exists by Exercise 1.6.29 (taking into account that morphism of affine schemes are the same as morphisms of locally ringed spaces, Proposition 1.6.13). $\square$

Proposition 1.7.23. Let $f : A \to B$ be an integral ring map. Then

$$\dim A \geqslant \dim B.$$

If f is an integral extension (i.e., injective), then

$$\dim A = \dim B.$$

Proof. To prove the first claim we take a chain of prime ideals in B and produce a chain of prime ideals in A of the same length. We reduce this claim to the case where A is a field: if $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ is a chain of prime ideals in B then the prime ideals $\mathfrak{p}_i = f^{-1}(\mathfrak{q}_i)$ form a chain in A. If, say, $\mathfrak{p} := \mathfrak{p}_i = \mathfrak{p}_{i+1}$, then $\mathfrak{q}_i \subsetneq \mathfrak{q}_{i+1}$ is a strict inclusion of prime ideals in $B_{\mathfrak{q}_{i+1}}/\mathfrak{q}_i B_{\mathfrak{q}_{i+1}}$. However, $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to B_{\mathfrak{q}_{i+1}}/\mathfrak{q}_i B_{\mathfrak{q}_{i+1}}$ is an integral map as well, so we have reduced to A being a field, say $A = k$.

We have reduced the first claim to showing this: if $k \to B$ is an integral map, then $\dim B \leqslant 0$. If $\operatorname{Spec} B \neq \emptyset$, pick a prime $\mathfrak{q} \subset B$. Then $C := B/\mathfrak{q}$ is a domain, and (as B) integral over k. We claim that C is a field. For $c \in C$, there is a (monic) polynomial $p(x) \in k[x]$ such that $p(c) = 0$. If we write $p(x) = \sum_i e_i x^i$, then we have $e_0 \neq 0$ (since C is a domain), therefore $c \cdot \frac{\sum_{i>0} -e_i c^{i-1}}{e_0} = 1$, so c is a unit in C.

If f is injective then φ is dominant (Lemma 1.1.12). Since it is also closed, φ is surjective. Given a specialization $x \rightsquigarrow x'$ in $\operatorname{Spec} A$ we can lift it to a specialization in $\operatorname{Spec} B$. (This was shown in the proof of (2) $\Rightarrow$ (3) above.) Thus $\dim B \geqslant \dim A$. □

Proposition 1.7.24. (*Noether normalization*) Let $A = k[t_1, \dots, t_n]/I$ be a finitely generated k-algebra with $\dim A = d$. Then there are d elements $a_1, \dots, a_d \in A$ such that the map

$$k[u_1, \dots, u_d] \to A, u_i \mapsto a_i$$

is injective and turns A into a finite $k[u_1, \dots, u_d]$-algebra. In other words, the map $\operatorname{Spec} A \to \mathbf{A}^d_k$ is finite.

Proof. See, e.g., [Stacks, Tag 00OY] or [Eis95, §8.2.1, Theorem A1] for a proof. □

Exercises

Exercise 1.7.25. Recall that a ring A is a *unique factorization domain* (abbreviation: UFD) if it is a domain and if any $x \in R$, $x \neq 0$, $x \notin R^\times$ is a (finite) product of irreducible elements, and any two such factorizations

$$x = a_1 \cdot \dots \cdot a_m = b_1 \cdot \dots \cdot b_n$$

implies $n = m$ and the a_i equal the b_j up to a permutation. (For example, $\mathbf{Z}$ and fields are UFDs; if A is a UFD, then so is $A[t]$; localizations of UFDs are again UFDs).

Prove that any UFD is integrally closed.

Exercise 1.7.26. Let A be a domain that is integrally closed in its fraction field $Q(A)$. Let $S \subset A$ be a submonoid (for the multiplication; also known as a multiplicatively closed subset), $0 \notin S$. Prove that the localization $A[S^{-1}]$ is an integrally closed domain as well.

Exercise 1.7.27. Let A be an integral domain. Prove that the following are equivalent:
(1) A is integrally closed,

(2) $A_\mathfrak{p}$ is integrally closed for all prime ideals $\mathfrak{p} \subset A$,

(3) $A_\mathfrak{m}$ is integrally closed for all maximal ideals $\mathfrak{m} \subset A$.

Hint: for (3) $\Rightarrow$ (1): if an element $s \in Q(A) \backslash A$ satisfies a monic equation, consider the ideal

$$I := \{r \in A \mid rs \in A\} \subsetneq A.$$

Exercise 1.7.28. Fix an inclusion of commutative rings $A \subset B$. Prove that B is a finite A-module iff B is integral over A and if it finitely generated as an A-algebra.

Exercise 1.7.29. Let $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ be a finite morphism, i.e., such that B is finite as an A-module. For any $\mathfrak{p} \in \operatorname{Spec} A$, prove that $\varphi^{-1}(\mathfrak{p})$ is homeomorphic to a disjoint union of finitely many copies of a singleton.

Exercise 1.7.30. Let $K \subset L$ be a field extension. Prove: the extension is algebraic (in the sense of field theory) iff it is integral. Use this to give an example of an integral, but not finite ring map $A \subset B$.

Exercise 1.7.31. Recall that the *support* of an A-module M is defined as

$$\operatorname{Supp} M := \{\mathfrak{p} \in \operatorname{Spec} A \mid M_\mathfrak{p} \neq 0\}.$$

We now suppose M is a finite A-module.

(1) Prove that
$$\operatorname{Supp} M = V(\operatorname{Ann} M),$$
where $\operatorname{Ann} M := \{a \in A \mid aM = 0\}$ is the *annihilator* of M. In particular, the support of M is closed.

(2) Also prove
$$\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \neq 0\}. \tag{1.7.32}$$
Hint: use the Nakayama lemma.

Exercise 1.7.33. Let $A \to B$ be a finite ring homomorphism. Use Exercise 1.7.31 to give another proof of the fact that $\operatorname{Spec} B \to \operatorname{Spec} A$ is closed.

Exercise 1.7.34. Let $A = k[x, y]/y^2 - x^3$ and $B = k[t]$, and consider the map $f : A \to B$ given by $x \mapsto t^2, y \mapsto t^3$. Prove the following

- f is injective, so that A is a domain.
- f is finite.
- A is not integrally closed in B, and therefore not integrally closed in $Q(A)$.
- $\dim A = 1$.
- Prove that the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is bijective (on the level of the underlying sets). Deduce that it is a homeomorphism (on the level of the underlying topological spaces).

  Hint: prove that the localization $A[x^{-1}] \to B[t^{-1}]$ is an isomorphism.
- Let A' be the localization of A at the prime ideal (x, y) and $\mathfrak{m} \subset A'$ its maximal ideal. Prove that the residue field $A'/\mathfrak{m}$ is isomorphic to k and prove $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$.

The ring A (or $\operatorname{Spec} A$) is referred to as the *cusp*. The map $\mathbf{A}^1_k = \operatorname{Spec} B \to \operatorname{Spec} A$ is referred to as the *normalization* of the cusp. Given that $\dim A' = 1 < \dim_k \mathfrak{m}/\mathfrak{m}^2$, the local ring A' is called *singular*.

1.8 Chevalley's theorem on constructible subsets

By design, all ring homomorphisms $A \to B$ induce continuous maps $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$; i.e., $\varphi^{-1}(U)$ is open for any open U. Images of open (resp. closed) subsets need in general not be open (resp. closed), as the following examples show.

Example 1.8.1. Consider the inclusion $A = \mathbf{Z}[x] \to B = \mathbf{Z}[x, y]/xy$. Geometrically $\varphi : \operatorname{Spec} \mathbf{Z}[x, y]/xy \to \mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}[x]$ corresponds to the projection of a coordinate cross to the x-axis. One checks that the image of $D(y)$ is the origin in $\mathbf{A}^1$, which in particular shows that the image of an open subset may be closed.

Example 1.8.2. Consider the canonical inclusion $\mathbf{Z}[x] \to \mathbf{Z}[x, y]$. Passing to spectra, we obtain the projection $\mathbf{A}^2 \to \mathbf{A}^1$. The closed subset $Z := V(xy - 1) \subset \mathbf{A}^2$ is the hyperbola. One checks that its image in $\mathbf{A}^1$ is $\varphi(Z) = \mathbf{A}^1_{\mathbf{Z}} \backslash \{0\} (= \operatorname{Spec} \mathbf{Z}[x, x^{-1}])$. In particular, the image of the closed subset Z is open.

These two examples suggest considering a combination of open and closed subsets.

Definition 1.8.3. A subset of $\operatorname{Spec} A$ is called *constructible* if it is a finite union of subsets of the form
$$D(f) \cap V(g_1, \dots, g_n),$$
for $f, g_1, \dots, g_n \in A$.

Remark 1.8.4. There is a definition of constructible subsets of arbitrary topological spaces [Stacks, Tag 005G]. The definition above agrees with that one by [Stacks, Tag 00F6].

Theorem 1.8.5. (*Chevalley's theorem on constructible sets*) Let B be a finitely presented A-algebra, i.e., $B = A[x_1, \dots, x_n]/(f_1, \dots, f_m)$ and consider the map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$. If $Z \subset \operatorname{Spec} B$ is constructible, then $\varphi(Z) \subset \operatorname{Spec} A$ is also constructible.

We defer the proof of this theorem to §1.10. To see that Chevalley's theorem imposes an actual restriction, here is an example of a non-constructible subset:

Lemma 1.8.6. Consider $\{\eta\} \subset \mathbf{A}^1_k$, where $\eta = (0)$ is the generic point. This subset is *not* constructible (for any field k).

Proof. If it was constructible, then its complement, which consists of all the closed points, would be constructible as well by Exercise 1.8.11, i.e., a *finite* union of closed points. However, there are infinitely many closed points (even if k is finite), as one sees by adapting Euclid's classical proof showing that there are infinitely many prime numbers: the closed points are of the form (f), with $f \in k[t]$ monic and irreducible polynomials. If there were only finitely many, say, $f_1, \dots, f_n$, then the irreducible factors of $\prod_{i=1}^n f_i + 1$ are distinct from the f_i, giving a contradiction. $\square$

Proposition 1.8.7. Let $f : A \to B$ be a finitely presented *flat* map. Then $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is an *open map* (i.e., $\varphi(U)$ is open for any open $U \subset \operatorname{Spec} B$).

Proof. [Stacks, Tag 00I1] This is basically a consequence of Chevalley's theorem (Theorem 1.8.5). It is enough to prove that $\varphi(D(b))$ is open, where $b \in B$. Since $D(b) \cong \operatorname{Spec} B[b^{-1}] \xrightarrow{j} \operatorname{Spec} B$ is flat and finitely presented, it is enough to show that $\varphi(\operatorname{Spec} B)$ is open. By Theorem 1.8.5 it is constructible. By Exercise 1.8.10 it is enough to show $\varphi(\operatorname{Spec} B)$ is stable under generization, i.e. given $\mathfrak{q} \in \operatorname{Spec} B$ and $\mathfrak{p} = \varphi(\mathfrak{q})$ and a generization $\mathfrak{p}' \rightsquigarrow \mathfrak{p}$, we need to find some $\mathfrak{q}' =?$ in the diagram below:

$$\begin{tikzcd} ? \arrow[r, rightsquigarrow] \arrow[d, mapsto] & \mathfrak{q} \arrow[d, mapsto] \\ \mathfrak{p}' \arrow[r, rightsquigarrow] & \mathfrak{p} \end{tikzcd}$$

We have the map $\operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} A_{\mathfrak{p}} = \{\mathfrak{p}', \mathfrak{p}' \rightsquigarrow \mathfrak{p}\}$. This is a local map of local rings. It is flat, being the composition of $A_{\mathfrak{p}} \to A_{\mathfrak{p}} \otimes_A B \to B_{\mathfrak{q}}$. Being a flat local map between local rings it is faithfully flat (Lemma 1.4.3). $\square$

Exercises

Exercise 1.8.8. Confirm the claims made about the images in Example 1.8.1 and Example 1.8.2.

Exercise 1.8.9. Prove the following converse to Chevalley's theorem: for any constructible subset $S \subset \operatorname{Spec} A$ there is a finitely presented A-algebra B such that $S = \operatorname{im}(\operatorname{Spec} B \to \operatorname{Spec} A)$.

Exercise 1.8.10. (Solution at p. 108) Let $S \subset \operatorname{Spec} A$ be a subset.

(1) Suppose S is open. Prove that S is stable under generization (i.e., for $x \in S$, $y \rightsquigarrow x$ one has $y \in S$).

(2) Conversely, prove that S is open provided that S is stable under generization and S is constructible.

Exercise 1.8.11. Show that complements and finite intersections of constructible subsets (inside $\operatorname{Spec} A$, for a fixed ring A) are again constructible.

The statements proved in the exercises Exercise 1.8.13 and Exercise 1.8.14 are key special cases of the openness of finitely presented flat maps (Proposition 1.8.7). We will use them in the proof of Chevalley's theorem, cf. p. 45. In a similar vein, Exercise 1.8.12 is a step in that proof.

Exercise 1.8.12. Let $A \to B$ be one of the following:
(1) $B = A/I$, for a finitely generated ideal $I \subset A$,

(2) $B = A[f^{-1}]$, for some $f \in A$.
Prove that the induced map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ preserves constructible subsets. I.e., if $S \subset \operatorname{Spec} B$ is constructible, prove that S is also constructible as a subset of $\operatorname{Spec} A$.

Hint for $B = A/I$: for $b \in B$, prove that $\operatorname{Spec} A \backslash D_{\operatorname{Spec} B}(b)$ is constructible and use Exercise 1.8.11.

Prove that the assertion fails for $B = A/I$ if I is not finitely generated: prove that the origin in the *infinite-dimensional affine space* $\mathbf{A}^\infty_k := \operatorname{Spec} k[t_1, t_2, \dots]$ is not a constructible subset.

Exercise 1.8.13. (Solution at p. 108) (Moret-Bailly `https://mathoverflow.net/q/481465`) Let $A \to B$ be a ring homomorphism such that B is a finitely generated free A-module, i.e., $B \cong A^d$ (as a module). Consider the map $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$.
(1) Let $b^d = (a_1, \dots, a_d)$ (in a basis of B). Prove that $\operatorname{Spec} A \backslash f(D(b)) = V(a_1, \dots, a_d)$.

(2) Deduce that φ is an open map.

Exercise 1.8.14. (Solution at p. 108) Let A be a ring, and consider the canonical map $\varphi : \mathbf{A}^1_A = \operatorname{Spec} A[x] \to \operatorname{Spec} A$.
(1) For any $f = \sum_{n=0}^d a_n x^n \in A[x]$, prove that $\varphi(D(f)) = \bigcup_n D(a_n) (\subset \operatorname{Spec} A)$.

(2) Deduce that φ is an open map.

Exercise 1.8.15. (Solution at p. 108) Let A be absolutely flat. Prove that a subset $S \subset \operatorname{Spec} A$ is constructible iff it is clopen (i.e., closed and open).

1.9 Hilbert's Nullstellensatz

Proposition 1.9.1. (*Hilbert's Nullstellensatz*) Let $A = k[t_1, \dots, t_n]/I$ be a finite type k-algebra, where k is a field. Let $x \in \operatorname{Spec} A$. Then x is a *closed* point (i.e., a maximal ideal) if and only if the residue field $k(x)$ is a *finite* extension of k.

There are many proofs of this theorem, for example [Eis95] contains five of them. The one below follows [MO15, Corollary II.2.11] and [Stacks, Tag 00FV].

Proof. We begin with the trivial direction "$\Leftarrow$": we have $k \subset A/\mathfrak{p} \subset (A/\mathfrak{p})_\mathfrak{p} = k(x)$, so if $\dim_k(A/\mathfrak{p}) \leqslant \dim_k k(x) < \infty$, so $A/\mathfrak{p}$ is a field. (Recall, as was shown in the proof of Proposition 1.7.23, that a domain that is also a finite-dimensional k-vector space is necessarily a field.) We therefore obtain that $\mathfrak{p}$ is maximal.

"$\Rightarrow$": We first treat the case $n = 1$ (and $I = 0$). In this case a closed point $x \in \mathbf{A}^1_k$ is generated by a non-zero monic irreducible polynomial $f \in k[x]$, and $k(x) = k[t]/f$ is indeed a finite extension of k.

For $n > 1$, consider the map

$$p_i : \operatorname{Spec} A \subset \mathbf{A}^n_k \xrightarrow{\mathrm{pr}_i} \mathbf{A}^1_k,$$

where the right hand map is the projection onto the i-th coordinate, i.e., on the level of rings given by

$$k[t_i] \subset k[t_1, \dots, t_n] \to A.$$

By Chevalley's theorem (Theorem 1.8.5), $p_i(x)$ is constructible, so, by Lemma 1.8.6 it is not the generic point, but a closed point. By the case $n = 1$, the residue field $k(p_i(x))$ is a finite extension of k. The field $k(x)$ is generated by the subfields $k(p_i(x))$ (since the $k[t_i]$ for $i \leqslant n$ generate $k[t_1, \dots, t_n]$ as a k-algebra). Thus, $k(x)$ is also finite over k. □

The following consequence explains why the above theorem is referred to as "Nullstellensatz" ("Null" = zero, "Stelle" = locus, "Satz" = theorem).

Corollary 1.9.2. If k is an algebraically closed field, the closed points of $\mathbf{A}^n_k = \operatorname{Spec} k[t_1, \dots, t_n]$ are in bijection to k^n. More precisely these are the prime ideals

$$(t_1 - a_1, \dots, t_n - a_n), \ a_i \in k.$$

More generally, the closed points of $V((f_1, \dots, f_m)) = \operatorname{Spec} k[t_1, \dots, t_n]/(f_1, \dots, f_m)$ are precisely the prime ideals above, where for all i

$$f_i(a_1, \dots, a_n) = 0.$$

Corollary 1.9.3. For a field k, $\dim \mathbf{A}^n_k = \dim k[t_1, \dots, t_n] = n$.

Proof. Assume first that k is algebraically closed. Let $\mathfrak{m} = (t_i - a_i, i \leqslant n)$ be a maximal ideal. We have the chain

$$\mathfrak{p}_0 = (0) \subsetneq \mathfrak{p}_1 = (t_1 - a_1) \subsetneq \cdots \subsetneq \mathfrak{p}_n = (t_1 - a_1, \dots, t_n - a_n) = \mathfrak{m}.$$

Thus, $\dim k[t_1, \dots, t_n] \geqslant n$. On the other hand, by Theorem 1.3.4(1), we have $\dim k[t_i]_{\mathfrak{m}} \leqslant n$, since $\mathfrak{m}$ is generated by n elements. Then $\dim k[t_i] = \sup_{\mathfrak{m} \text{ maximal}} \dim k[t_i]_{\mathfrak{m}} = n$.

For a general field k, the claim follows from Lemma 1.9.4. Alternatively, one can use a similar, but more elaborate argument along the lines above [Stacks, Tag 00OP]. □

Lemma 1.9.4. Let k be a field, k' an algebraic field extension (but not necessarily finite, so $k' = \overline{k}$ is allowed) and A a k-algebra. Then $\dim A = \dim A \otimes_k \overline{k}$.

This statement fails if k' is transcendental over k, cf. Exercise 1.9.7.

Proof. The map $k \to k'$ is integral. By Theorem 1.7.20 so is its base change $A \to A \otimes_k k'$. The map is injective since A is flat over k (Example 1.4.1). Thus their dimensions agree (Proposition 1.7.23). □

Exercises

Exercise 1.9.5. Let k be an algebraically closed field.

(1) For $i = 1, 2$, let A_i be a finitely generated k-algebra. Let $X_i := \operatorname{Spec} A_i$ and write $X := \operatorname{Spec}(A_1 \otimes_k A_2)$. (In the language of Proposition 2.4.2, this is the fiber product $X = \operatorname{Spec} A_1 \times_{\operatorname{Spec} k} \operatorname{Spec} A_2$.) Prove that there is a bijection

$$X^{\mathrm{cl}} = X_1^{\mathrm{cl}} \times X_2^{\mathrm{cl}},$$

where the superscript cl denotes the set of closed points (and at the right we have the products of these two *sets*).

(2) Exhibit (non-closed) points in $\mathbf{A}^2_k = \mathbf{A}^1_k \times_{\operatorname{Spec} k} \mathbf{A}^1_k$ that are not pairs of points in the two copies of $\mathbf{A}^1_k$.

(3) Show that the assumption of k being algebraically closed cannot be dropped: prove that $\operatorname{Spec}(\mathbf{Q}[\sqrt{2}] \otimes_{\mathbf{Q}} \mathbf{Q}[\sqrt{2}])$ consists of two points, for example.

(4) (bonus, for those with knowledge of Galois theory): For an arbitrary field k and a separable closure k^{sep}, establish a homeomorphism where at the left we have the absolute Galois group

$$\operatorname{Gal}(k^{\mathrm{sep}}/k) \xrightarrow{\cong} \operatorname{Spec}(k^{\mathrm{sep}} \otimes_k k^{\mathrm{sep}}).$$

(Hint: first prove the statement if k^{sep} is replaced by a finite Galois extension.)

Exercise 1.9.6. (Solution at p. 109) In Corollary 1.9.2, why is it necessary to require k to be algebraically closed?

Exercise 1.9.7. Let k be a field. Prove that (in contrast to Lemma 1.9.4)

$$\dim(k(t) \otimes_k k(u)) = 1.$$

Hint: $\operatorname{Spec} k(t)$ is the generic point of $\mathbf{A}^1_k = \operatorname{Spec} k[t]$. Compare $\operatorname{Spec}(k(t) \otimes_k k(u))$ with $\operatorname{Spec}(k[t] \otimes_k k[u])$.

Exercise 1.9.8. Let $x \in \mathbf{A}^n_{\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[t_1, \dots, t_n]$. Show that x is a closed point if and only if its residue field $k(x)$ is a finite field.

1.10 Proof of Chevalley's theorem

In this section, we prove Chevalley's Theorem 1.8.5 with the method due to Olivier [Oli78]. Given the map $\operatorname{Spec} B \to \operatorname{Spec} A$, the proof will proceed as follows:

(1) A series of relativy easy reduction steps shows that it is enough to consider $B = A[t]$, i.e., geometrically the projection $\mathbf{A}^1_A \to \operatorname{Spec} A$.

(2) We will reduce to the case of A being an absolutely flat ring (Definition 1.4.8). Roughly speaking, this amounts to tearing $\operatorname{Spec} A$ apart. This reduction step will use as an input that Chevalley's theorem holds true for maps of the form

$$D(a) \sqcup V(a) \to \operatorname{Spec} A,$$

cf. Exercise 1.8.12.

(3) In the case when A is absolutely flat, we will be able to inspect the statement basically by hand. As it turns it is then enough to show that Chevalley's theorem holds for maps of the following two types:

- $\operatorname{Spec} B \to \operatorname{Spec} A$, where B is a finite free A-module (Exercise 1.8.13),
- $\mathbf{A}^1_A \to \operatorname{Spec} A$ (cf. Exercise 1.8.14).

A category-theoretic interlude

The following category-theoretic statement is known as the *small object argument*. It plays an outsize rôle in homotopy theory, a branch of algebraic topology. It can also be used to systematize various ring-theoretic constructions in algebraic geometry. We will employ it in order to perform the reduction step (2) alluded to above.

Lemma 1.10.1. Let C be a compactly generated category that admits all (small) colimits. Let $g_i : A_i \to B_i$ be a set of maps (indexed by $i \in I$), and assume that the objects A_i are compact. Then, for any map $f : X \to Z$ in C, there is a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f'"'] & & Z \\ & Y \arrow[ur, "f''"'] & \end{tikzcd}$$

where f'' satisfies the right lifting property with respect to the maps g_i and f' lies in the *saturation* of the set $\{g_i\}$, i.e., it is obtained from the maps g_i by taking coproducts, pushouts and transfinite compositions.

Proof. See, e.g. [Lur09, Proposition A.1.2.5]. The heuristic idea of the proof is simple: consider the question whether f satisfies the right lifting property relative to the g_i:

$$\begin{tikzcd} A_i \arrow[r, "h"] \arrow[d, "g_i"'] & X \arrow[d, "f"] \\ B_i \arrow[r] \arrow[ur, dotted, "?"] & Z \end{tikzcd}$$

If these lifts do exist, there is nothing to be done. To account for the possibility that such a lift does not exist, consider the set, denoted $A(f)$, of commutative squares as above (without the dotted arrow). Note this is a set, since the g_i are indexed by a set and since morphisms between any two fixed objects also form a set. Consider the factorization

$$\begin{tikzcd} \bigsqcup_{a\in A(f)} A_i \arrow[r] \arrow[d] & X \arrow[d, "f_1'"] \\ \bigsqcup_{a\in A(f)} B_i \arrow[r] & X_1 := X \sqcup_{\bigsqcup A_i} \bigsqcup B_i \arrow[d, "v_1"] \\ & Z. \end{tikzcd}$$

By construction, f_1 is a pushout of a coproduct of maps g_i, i.e., in the saturation. The map v_1, however, may not satisfy the right lifting property relatively to the g_i. (It does satisfy it if we only allow maps $A_i \to X'$ factoring over X as in the diagram above, though.) We can repeat the construction with v_1 in place of f, and construct a refined factorization

$$X \to X_1 \to X_2 \to Z.$$

Repeating this (countably many times) and setting $Y := \operatorname{colim} X_n$ gives a factorization $X \stackrel{f'}{\to} Y \stackrel{f''}{\to} Z$. By construction, f' is a transfinite composition of maps in the saturation of the g_i. We check that f'' satisfies the right lifting property:

$$\begin{tikzcd} A_i \arrow[r, dotted] \arrow[d, "g_i"'] \arrow[rrr, bend left] & X_n \arrow[r] & X_{n+1} \arrow[r] & Y \arrow[d, "f''"] \\ B_i \arrow[urr, dotted] \arrow[rrr] & & & Z \end{tikzcd}$$

Since A_i is compact, the map $A_i \to Y = \operatorname{colim} X_n$ factors over some X_n, as shown. (This step is why the lemma is called *small* object argument). By construction, the map $A_i \to X_n \to X_{n+1}$ factors through g_i as shown, in such a way that the top left triangle and the bottom right part commutes.

Lemma 1.10.2. If the maps g_i above are epimorphisms, then the factorization above is unique up to unique isomorphism. That is, given two factorizations of $f = f_1'' \circ f_1' = f_2'' \circ f_2'$, there is a unique isomorphism $Y \to Y'$ making the entire diagram commutative:

$$\begin{tikzcd} X \arrow[r, "f_1'"] \arrow[d, "f_2'"'] & Y' \arrow[d, "f_1''"] \\ Y \arrow[r, "f_2''"'] \arrow[ur, dotted, "\cong", "\varphi"'] & Z \end{tikzcd}$$

Proof. First, there is some morphism $\varphi : Y' \to Y$ making the diagram commute since f_1'' satisfies the right lifting property against the maps g_i and therefore also against the maps in the saturation of the g_i. Similarly, there is a map $\psi : Y \to Y'$ going the other way (not depicted). It suffices to check their composites are the identity maps, which reduces us to considering in the above diagram the case $Y' = Y$, and the composite $\psi \circ \varphi$ and id_Y. The composite of these two maps with f_1' is the same. But any map in the saturation of the g_i is an epimorphism, since the g_i are. Hence $\psi \circ \varphi = \mathrm{id}_Y$ and likewise for $\varphi \circ \psi$. □

Example 1.10.3. For any fixed ring A, the small object argument is applicable to the category $C = \mathrm{Alg}_A$ (of A-algebras and A-algebra morphisms). Indeed, it has all colimits: by general theory, it is enough to check the existence of pushouts and filtered colimits. Pushouts $R \sqcup_S T$ are precisely the tensor products $R \otimes_S T$. The filtered colimit $\mathrm{colim}_i R_i$ of a system of A-algebras is just the ring $R := \bigsqcup_i R_i / \sim$, where $R_i \ni r_i \sim r_j \in R_j$ iff there is some $k > i$, $k > j$ such that r_i and r_j map to the same element in R_k (under the transition maps $R_i \to R_k \leftarrow R_j$). The addition and multiplication are defined in the natural manner (which is well-defined since the transition maps are ring homomorphisms.)

Any A-algebra R is a filtered colimit of finitely presented A-algebras. Indeed, R is first the filtered colimit of its finitely generated A-algebras, namely the subalgebras of R generated by finitely many elements $r_1, \dots, r_n$. These subalgebras might not be finitely presented, but a finitely generated algebra $R = A[t_1, \dots, t_n]/(f_i, i \in I)$ is the filtered colimit of the finitely presented algebras $A[t_1, \dots, t_n]/(f_i, i \in J)$ for increasingly large finite subsets $J \subset I$.

An A-algebra R is a compact object in Alg_A iff it is a finitely presented A-algebra.

The weak saturation of $0 \sqcup \mathbf{G}_{\mathrm{m}} \subset \mathbf{A}^1$

In order to get mileage out of the small object argument, one needs to understand a) the saturation of a set of maps and b) the maps satisfying the lifting property relative to those maps. To warm up for Olivier's proof of Chevalley's theorem, we consider a slightly more basic example first.

Lemma 1.10.4. Consider the map $g : \mathbf{Z}[t] \to \mathbf{Z}[t, t^{-1}]$. The saturation of this map consists precisely of the ring homomorphisms of the form $A \to A[S^{-1}]$, i.e., localizations.

A map $A \to B$ satisfies the right lifting condition

$$\begin{tikzcd} \mathbf{Z}[t] \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ \mathbf{Z}[t^{\pm 1}] \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

iff it is *conservative*, i.e., if for any element $a \in A$ such that $f(a) \in B^\times$ we already have $a \in A^\times$. (If A and B are local rings, this is precisely the condition of being a local map in the sense of Definition and Lemma 1.2.3).

Proof. Indeed, $A[S^{-1}] = \mathrm{colim}\, A[\{s_1, \dots, s_n\}^{-1}]$, and the terms in the colimit agree with $\bigotimes_{i=1}^n A[s_i^{-1}]$. Hence we are reduced to observing the following pushout diagram:

$$\begin{tikzcd} \mathbf{Z}[t] \arrow[r, "t \mapsto s"] \arrow[d] & A \arrow[d] \\ \mathbf{Z}[t^{\pm 1}] \arrow[r] & A[s^{-1}]. \end{tikzcd}$$

Conversely, the same reasoning shows that any morphism in the saturation of g is a localization.

The second statement is clear by $\mathrm{Hom}_{\mathrm{Rings}}(\mathbf{Z}[t^{\pm 1}], A) = A^\times$ etc. □

The map g is an epimorphism (even though it is not surjective!), so Lemma 1.10.2 supplies a unique factorization of any ring homomorphism $A \to B$ as

$$f = f'' \circ f' : A \xrightarrow{f'} A[S^{-1}] \xrightarrow{f''} B$$

with f' being a localization and f'' being conservative. The latter can be made concrete by observing that we have a factorization

$$A \to A[(f^{-1}(B^\times))^{-1}] \to B$$

where the second map is conservative. By unicity (up to unique isomorphism) of the factorization (Lemma 1.10.2), we see that the abstractly supplied factorization is this one.

A subexample of the above: if $B = k(\mathfrak{p})$ is the residue field of a prime ideal $\mathfrak{p} \subset A$, this factorization is

$$A \to A_\mathfrak{p} \to k(\mathfrak{p}).$$

Lemma 1.10.5. Let $g : \mathbf{Z}[t] \to \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}]$ be given by $t \mapsto (0, t)$ and write $\varphi : \operatorname{Spec} \mathbf{Z} \sqcup \mathbf{G}_\mathrm{m} \to \mathbf{A}^1$ for the induced map on spectra. Let g' be a map in the saturation of g. Then

- g' is bijective on the level of the maps of spectra (we do not assert these maps are homeomorphisms of the underlying topological spaces).
- g' preserves constructible subsets.

Proof. Indeed, the bijectivity holds true for φ and finite products $\varphi^n : (\operatorname{Spec} \mathbf{Z} \sqcup \mathbf{G}_\mathrm{m})^{\times n} \to \mathbf{A}^n$ and also for any pullback of such maps:

$$\begin{array}{ccc} \operatorname{Spec} A' & \longrightarrow & (\operatorname{Spec} \mathbf{Z} \sqcup \mathbf{G}_\mathrm{m})^{\times n} \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle \varphi^n} \\ \operatorname{Spec} A & \xrightarrow{(f_1,\dots,f_n)} & \mathbf{A}^n \end{array}$$

Indeed, $\operatorname{Spec} A'$ is the disjoint union of spectra of $\operatorname{Spec} A/(f_i, i \in I) \cap \operatorname{Spec} A[f_i^{-1}, i \notin I]$, where I runs through the (finite) subsets of $\{1, \dots, n\}$. This is nothing but $V(f_i, i \in I) \cap D(\prod f_i, i \notin I)$, and the union of these is $\operatorname{Spec} A$. Similarly, g' as above preserves constructible subsets by Exercise 1.8.12.

We claim that for a diagram of rings (indexed by $n \in \mathbf{N}$)

$$A_0 \to A_1 \to \dots \to A_\infty := \operatorname{colim} A_n$$

with each map inducing a bijection $\operatorname{Spec} A_{n+1} \to \operatorname{Spec} A_n$, the map $\operatorname{Spec} A_\infty \to \operatorname{Spec} A_0$ also is a bijection. (Similarly, if each of these former maps preserves constructible subsets, then so does the latter.) This will imply the lemma.

We use that as a set $\operatorname{Spec} A$ consists of maps $A \to k$, for fields k, up to the identification $(f : A \to k) \sim (f' : A \to k')$ iff there is a field k'' containing k and k' such that $f = f'$ when regarded as maps $A \to k''$ (Exercise 1.2.9). Given a point in $\operatorname{Spec} A_0$, i.e., a map $A_0 \to k_0$, there is a field extension k_1/k_0 such that $A_0 \to k_0 \subset k_1$ factors through A_1 as shown below:

$$\begin{array}{ccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & \dots & \longrightarrow & A_\infty = \operatorname{colim} A_i \\ \downarrow & & \downarrow & & & & \downarrow \\ k_0 & \hookrightarrow & k_1 & \hookrightarrow & \dots & \hookrightarrow & \bigcup k_i \end{array}$$

The filtered colimit of these fields, $\bigcup_i k_i$, is also a field and this gives a point in $\operatorname{Spec} A_\infty$; showing the surjectivity of $\operatorname{Spec} A_\infty \to \operatorname{Spec} A_0$. The injectivity is similar: given two points $\operatorname{colim} A_i \rightrightarrows k$ whose composite with A_0 agree, then its restriction to A_1 agrees etc.

The argument about constructibility is similar, since constructible subsets in $\operatorname{Spec} A_\infty$ are finite unions of subsets of the form $V(f_1, \dots, f_n) \cap D(f)$. Then one uses that the elements $f, f_i \in A_\infty$ arise from some A_n. $\square$

The proof

Proof of Theorem 1.8.5. We begin with a (standard) reduction step: it is enough to prove the theorem for $B = A[x]$. Indeed, a constructible subset in $\operatorname{Spec} B = \operatorname{Spec} A[x_1, \dots, x_n]/(f_1, \dots, f_m)$ is also constructible in $\operatorname{Spec} A[x_1, \dots, x_n]$ (Exercise 1.8.12; this step uses that B is finitely *presented*, as opposed to finitely generated over A), so we may assume $B = A[x_i]$. Then by induction, it suffices to consider a single variable.

By the definition of constructible sets, we have to consider a finitely generated ideal $I \subset B$, and $f \in B$ and show that the image of $W := V(I) \cap D(f)$ in $\operatorname{Spec} A$ is constructible.

We apply the small object argument (Lemma 1.10.2) to the map $g : 0 \sqcup \mathbf{G}_{\mathrm{m}} \to \mathbf{A}^1$ (which is a monomorphism of affine schemes, cf. Exercise 1.4.17(2)). The map $\varnothing \to \operatorname{Spec} A$ thus factors uniquely (up to unique isomorphism) as

$$\varnothing \to \operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A,$$

where the map $\varnothing \to \operatorname{Spec} \widetilde{A}$ satisfies the (unique) left lifting property against g. In other words, $\widetilde{A}$ is absolutely flat by Lemma 1.4.9.

Consider the absolutely flat ring $\widetilde{A}$ and the associated pullback diagram

$$\begin{tikzcd}
\widetilde{W} \arrow[r] \arrow[d, "g"] & \operatorname{Spec} \widetilde{A}[t]/I \arrow[r, "\tilde{i}"] \arrow[d] & \mathbf{A}^1_{\widetilde{A}} \arrow[r, "\tilde{\pi}"] \arrow[d] & \operatorname{Spec} \widetilde{A} \arrow[d, "f"] \\
W \arrow[r] & \operatorname{Spec} A[t]/I \arrow[r, "i"] & \mathbf{A}^1_A \arrow[r, "\pi"] & \operatorname{Spec} A.
\end{tikzcd}$$

By Lemma 1.10.5, the map f and all the vertical maps in the above diagram are bijections. Thus the image of W in $\operatorname{Spec} A$ is the image of W' in $\operatorname{Spec} \widetilde{A}$. Also by Lemma 1.10.5, f preserves constructible subsets.

We can therefore replace A by $\widetilde{A}$ in the sequel and assume A is absolutely flat. We claim that for our absolutely flat ring A, and $I \subset A[t]$ finitely generated, $\operatorname{Spec} A$ decomposes into a disjoint union of $\operatorname{Spec} A = \bigsqcup_{i=1}^n \operatorname{Spec} A_i$, such that the fibers of $\pi \circ i$ are either $\mathbf{A}^1_{A_i}$ or $V(g_i) \subset \mathbf{A}^1_{A_i}$, where $g_i \in A_i[t]$ is a monic polynomial. Indeed, for any $\mathfrak{p} \in \operatorname{Spec} A$, the $I \otimes_A k(\mathfrak{p})$ is (by absolute flatness of A) a submodule (i.e., an ideal) of $k(\mathfrak{p})[t]$. It is generated by a monic polynomial (here we consider 0 to be a monic polynomial as well), say f. There is a some fundamental open neighborhood $D(b) \ni \mathfrak{p}$ such that f extends to an element in $A[b^{-1}][t]$, for some $b \in A$. By Exercise 1.8.15, the constructible subsets are the clopen subsets in $\operatorname{Spec} A$, so the claim is true on a clopen neighborhood of any point, as requested.

Using that claim, it suffices to check the constructibility claim for $I = 0$ and for $I = (g)$ with g monic:

- The former case holds by Exercise 1.8.14 (for *any* ring E, the map $\mathbf{A}^1_E \to \operatorname{Spec} E$ is open).
- For the latter case we use that for *any* ring E, and any monic polynomial $g \in E[t]$, then $F := E[t]/g$ is finite free as an E-module, so that by Exercise 1.8.13, the image of $D(f) \subset \operatorname{Spec} F$ is constructible for any $f \in F$. □

Exercises

Exercise 1.10.6. Let A be a discrete valuation ring with residue field k and quotient field K. (If you prefer taking a more concrete example you can pick $A = \mathbf{Z}_p$ with $k = \mathbf{F}_p$ and $K = \mathbf{Q}_p$ or alternatively $A = k[t]_{(t)}$, with residue field being k and $K = k(t)$.) Show that the absolutely flat ring constructed in the proof above is given by

$$\widetilde{A} = k \times K.$$

Hint: show that a proper field extension $E \subsetneq F$ never induces a bijection $\mathbf{A}^1_F \to \mathbf{A}^1_E$. I.e., $E \to F$ is not in the weak saturation of the map $0 \sqcup \mathbf{G}_{\mathrm{m}} \to \mathbf{A}^1$.

Exercise 1.10.7. With A a discrete valuation ring, use Exercise 1.10.6 to illustrate all the steps in Olivier's proof of Chevalley's theorem for $B = A[t]/t^2 - \varpi$, where ϖ is a uniformizer, i.e., a generator of the maximal ideal $\mathfrak{m} \subset A$.

Chapter 2

Schemes

Definition 2.0.1. A *scheme* is a locally ringed space that locally looks like an affine scheme. More formally, it is a locally ringed space $(X, \mathcal{O}_X)$ such that for every point $x \in X$ there is a (commutative) ring A (depending on x) and an open neighborhood U and an isomorphism (of locally ringed) spaces

$$(U, \mathcal{O}_U) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$

(Here $(U, \mathcal{O}_U) := (U, \mathcal{O}_X|_U)$ carries the induced structure of a locally ringed space, cf. Exercise 1.6.29). The full subcategory of LocRingedSpace consisting of schemes is denoted by Sch.

To simplify the notation, we will usually only denote a scheme by X, leaving the structural sheaf $\mathcal{O}_X$ implicit.

Of course, by definition, any affine scheme is a scheme in this sense, so that we have an inclusion of full subcategories

$$\text{Rings}^{\text{op}} \cong \text{AffSch} \subset \text{Sch} \subset \text{LocRingedSpace}.$$

Definition 2.0.2. If S is a scheme, the category of S-schemes Sch_S is the overcategory of $S \in \text{Sch}$. That is, objects of Sch_S are morphisms of schemes $X \to S$, and morphisms in Sch_S are commutative triangles

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

We refer to objects in Sch_S also as "schemes over S".

This definition is often applied when $S = \operatorname{Spec} A$ is affine. Given some $X \in \text{Sch}_S$, fix any open affine covering $X = \bigcup U_i$ by affines $U_i = \operatorname{Spec} B_i$. Then B_i are A-algebras. Similarly, morphisms $X \to Y$ over S are locally of the form $\operatorname{Spec} B \to \operatorname{Spec} C$, with $C \to B$ being an A-algebra map.

Exercises

In the following exercises we use the following notation for a scheme X and a ring A:

$$X(A) := \operatorname{Hom}_{\text{Sch}}(\operatorname{Spec} A, X).$$

We refer to the set $X(A)$ as the set of *A-valued points*. (If $X = \operatorname{Spec} B$ is affine, then we have

$$X(A) = \operatorname{Hom}_{\text{Rings}}(B, A).$$

Even more specifically, if $B = \mathbf{Z}[t_1, \dots, t_n]/(f_1, \dots, f_m)$ we have

$$X(A) = \{(a_1, \dots, a_n) \mid f_i(a_1, \dots, a_m) = 0\}.$$

In other words, the set $X(A)$ encodes the solutions in the given ring A of the system of polynomial equations. Of course, this depends dramatically on the ring A. The famous Fermat curve

$$x^n + y^n = 1$$

has certainly infinitely many solutions in $\mathbf{R}$, say, but according to the so-called *Fermat's last theorem* (proved by Wiles in 1995) the only solutions in $\mathbf{Z}$ for $n \geqslant 3$ are the "trivial" solutions, where at least one of the three variables is 0. Going back to the case of a general scheme we retain the observation that X determines the sets $X(A)$ for *all* rings A, i.e., X "knows" about the solutions of polynomial equations in all rings at the same time.)

Exercise 2.0.3. Let X be a scheme. Construct a bijection between the set X (i.e., disregarding the topology and the structural sheaf) and the set

$$\bigsqcup_k \operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec} k, X)/\sim,$$

where the coproduct runs over all fields k, and we identify $f : \operatorname{Spec} k \to X$ with $g : \operatorname{Spec} k' \to X$ (for another field k') iff there is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "f"] \arrow[d] & X \\ \operatorname{Spec} k' \arrow[ur, "g"] & \end{tikzcd}$$

In particular, deduce a bijection for the k-points

$$X(k) = \{x \in X, k(x) \to k\}.$$

Hint: reduce to the assertion in Exercise 1.2.9.

Exercise 2.0.4. Let A be a local ring and X a scheme.
(1) Let $f : \operatorname{Spec} A \to X$ be a morphism of schemes. Prove that its set-theoretic image $f(\operatorname{Spec} A)$ is contained in any affine neighborhood $U \subset X$ of the point $f(\mathfrak{m}_A)$, where $\mathfrak{m}_A$ is the unique maximal ideal of A.

(2) Deduce the following description of the A-points of X:

$$X(A) = \{x \in X, \mathcal{O}_{X,x} \to A (\text{local map of local rings})\}.$$

Exercise 2.0.5. Let X be a quasi-compact scheme (i.e., its underlying topological space satisfies the condition in Definition 1.1.7). Prove that any non-empty closed subset $Z \subset X$ contains closed point of X. In particular, X itself has a closed point. (This statement fails if X is not quasi-compact, see [Liu02, Exercise 3.27] for a counter-example of the form $X = \operatorname{Spec} V \backslash \{\mathfrak{m}_V\}$ where V is a certain (non-Noetherian) valuation ring.)

2.1 Open subschemes and glueing

Lemma 2.1.1. Let $(X, \mathcal{O}_X)$ be a scheme and $U \subset X$ an open subset of the topological space X. Then $(U, \mathcal{O}_X|_U)$ is a scheme as well.

Proof. Pick a covering $X = \bigcup_{i \in I} X_i$, with $X_i = \operatorname{Spec} A_i$ being affine. For all $x \in U$, pick some $i \in I$ such that $x \in X_i$. Then $U \cap X_i$ is an open neighborhood of x in the affine scheme X_i, so there is some $a \in A_i$ such that $x \in D_{\operatorname{Spec} A_i}(a) \subset U \cap X_i$. Note that $D(a)$ is an affine scheme, so we have produced an open affine neighborhood of x in U. $\square$

Despite its simplicity, the statement is not completely harmless. More precisely, the statement would fail if we were to replace "scheme" by "affine scheme", as the following example shows.

Example 2.1.2. Let $X = \mathbf{A}^2$ (or, in the same vein, $\mathbf{A}^n$ for $n \geqslant 2$) and consider the punctured plane $U = \mathbf{A}^2 \backslash \{(0,0)\}$, where we remove the origin, i.e., the closed point given by the maximal ideal (t_1, t_2). We claim that U is *not* an affine scheme. Indeed, by Exercise 1.6.32, we have $\mathcal{O}_U(U) = \mathbf{Z}[t_1, t_2]$. Given the equivalence of categories (1.6.19), the natural map

$$U \to \operatorname{Spec}(\mathcal{O}_U(U)) = \operatorname{Spec} \mathbf{Z}[t_1, t_2] = \mathbf{A}^2$$

would be an isomorphism. This map is the canonical inclusion $U \subset X$, which however is not an isomorphism since it is not bijective on the level of the underlying topological spaces.

The scheme U is an example of a *quasi-affine scheme*, i.e., an open subscheme of an affine scheme.

Recall from topology the glueing of topological spaces: given a (possibly infinite) family of topological spaces X_i, $i \in I$ and open subsets $X_{ij} \subset X_i$ (for each $j \in I$), where $X_{ii} = X_i$, and homeomorphisms

$$\varphi_{ij} : X_{ij} \stackrel{\cong}{\to} X_{ji}$$

satisfying the so-called *cocycle condition*

$$\varphi_{jk}|_{X_{ji} \cap X_{jk}} \circ \varphi_{ij}|_{X_{ij} \cap X_{ik}} \quad = \quad \varphi_{ik}|_{X_{ij} \cap X_{ik}}$$

there is a unique topological space X that is glued together from the X_i and the above data, namely

$$X := \bigsqcup_i X_i / \sim,$$

where the relation is the equivalence relation generated by identifying, for any $x_{ij} \in X_{ij}$, $x_{ij} (\in X_i)$ with $\varphi_{ij}(x_i j) \in X_{ji}$. In addition, we have the following universal property of X: for any topological space Y, we have

$$\operatorname{Hom}_{\mathrm{Top}}(X, Y) = \{(f_i : X_i \to Y) | f_i|_{X_{ij}} = f_j|_{X_{ji}} \circ \varphi_{ij}\}. \tag{2.1.3}$$

In categorical terms,

$$X = \operatorname{colim}\left(\bigsqcup_{i,j} X_{ij} \rightrightarrows \bigsqcup_i X_i \right). \tag{2.1.4}$$

The following statement is referred to by saying that *schemes glue* (along open subschemes).

Lemma 2.1.5. Using the above notation, assume that each X_i is a scheme, and the isomorphisms φ_{ij} are isomorphisms of schemes. Then X is naturally a scheme in such a way that for any scheme Y, the formula (2.1.3) (with morphisms of schemes) holds.

Proof. We endow the topological space X discussed above with the structural sheaf $\mathcal{O}_X$ constructed in Exercise 1.5.10 (given the isomorphisms $\mathcal{O}_{X_i}|_{X_{ij}} \cong \varphi_{ji,*}\mathcal{O}_{X_j}|_{X_{ji}}$ etc. that are part of the isomorphism of schemes φ_{ij}). The resulting pair $(X, \mathcal{O}_X)$ is a locally ringed space; note that the stalks $\mathcal{O}_{X,x}$ for $x \in X$ identify with $\mathcal{O}_{X_i,x}$, provided $x \in X_i$. Finally, any $x \in X_i (\subset X)$ has an open affine neighborhood inside X_i, and therefore also inside X. □

Remark 2.1.6. Lemma 2.1.5 says that (2.1.4) holds verbatim for schemes. In topology, one shows that *any* diagram of topological spaces admits a colimit. By contrast, more general colimits in the category of schemes do not usually exist.

The next statement, called *affine communication lemma* in [Vak17, p. 5.3.2] will help us organize various local-to-global arguments (i.e., extending statements from affine schemes to arbitrary schemes).

Proposition 2.1.7. Let P be a property of affine schemes (or, equivalently, of rings). We write $P(U)$ if P holds for U. Suppose:

(1) $P(\operatorname{Spec} A)$ (for $U = \operatorname{Spec} A \subset X$) implies $P(\operatorname{Spec} A[f^{-1}])$,

(2) if $U = \operatorname{Spec} A(\subset X)$ is covered by $U_i := \operatorname{Spec} A[f_i^{-1}]$, for $f_i \in A, i = 1, \dots, n$, then $P(U_i)$ for all i implies $P(U)$.

We call such a property *affine-local*. Now, if $X = \bigcup \operatorname{Spec} A_i$ with $P(\operatorname{Spec} A_i)$, then $P(U)$ holds for *any* open affine subscheme $U \subset X$.

To prove this statement, we will use the following argument about "well-placed" basic open neighborhoods.

Lemma 2.1.8. If $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are open affine subschemes of a scheme X, then $\operatorname{Spec} A \cap \operatorname{Spec} B$ is a union of open subsets that are at the same time basic open subsets (in the sense of Definition 1.1.1) inside $\operatorname{Spec} A$ and also inside $\operatorname{Spec} B$.

Proof. Pick a point $x \in \operatorname{Spec} A \cap \operatorname{Spec} B$. We can find a basic open subset $D_A(f) = \operatorname{Spec} A[f^{-1}]$ that is contained in $\operatorname{Spec} A \cap \operatorname{Spec} B$, and that contains x. Let $D_B(g) = \operatorname{Spec} B[g^{-1}]$ be a basic open subset contained in $D_A(f)$ and containing x. We have the restriction map $B = \mathcal{O}_X(\operatorname{Spec} B) \to \mathcal{O}_X(\operatorname{Spec} A[f^{-1}]) = A[f^{-1}]$, and we denote by $g' = \frac{g''}{f^n}$ the image of g under that map (with $g'' \in A$). We have

$$D_B(g) = \operatorname{Spec} B[g^{-1}] = \{\mathfrak{p} \in \operatorname{Spec} A[f^{-1}], g' \notin \mathfrak{p}\} = \operatorname{Spec}(A[f^{-1}])[g'^{-1}] = \operatorname{Spec} A[(fg'')^{-1}].$$

This is therefore a basic open subset in both $\operatorname{Spec} B$ and $\operatorname{Spec} A$ containing x. □

Proof of Proposition 2.1.7. Let $\operatorname{Spec} A \subset X$. By Lemma 2.1.8 and the quasi-compactness of $\operatorname{Spec} A$ (Lemma 1.1.10), we can find a finite covering of $\operatorname{Spec} A$ by basic open subsets $\operatorname{Spec} A[g_i^{-1}]$ which are also basic open subsets of $\operatorname{Spec} A_i$. Then, using our two assumptions on the property P:

$$P(A_i) \Rightarrow P(A[g_i^{-1}]) \ \forall i \Rightarrow P(A).$$

A quick way to obtain an affine-local property of schemes is to demand some property of the stalks at all points inside the given affine subset.

Definition 2.1.9. A scheme X is called *normal* (resp. *factorial*) if all the stalks $\mathcal{O}_{X,x}$ are integrally closed domains (resp. unique factorization domains).

By Exercise 1.7.27, an integral scheme X is normal iff all $\mathcal{O}_X(U)$ (for affine $U \subset X$) is integrally closed. By Exercise 1.7.25, any factorial scheme is normal. The converse does not hold: an example studied in number theory is $\operatorname{Spec} \mathcal{O}_K$, the integral closure of $\mathbf{Z}$ inside a number field K, i.e., a finite extension $K/\mathbf{Q}$. This is always normal, but not necessarily a UFD. A prototypical example from number theory is $\mathbf{Z}[\sqrt{-5}]$.

Definition 2.1.10. A scheme X is called *locally Noetherian* if it admits an open covering $X = \bigcup_i U_i$ with $U_i = \operatorname{Spec} A_i$ and A_i is a Noetherian ring. X is called *Noetherian* if it is locally Noetherian and quasi-compact (Definition 1.1.7).

Again, this definition is sensible in view of the fact (proved in Exercise 2.1.11) that being Noetherian is an affine-local property. Thus, X is locally Noetherian iff for any open affine $\operatorname{Spec} A \subset X$, A is Noetherian.

Exercises

Exercise 2.1.11. Prove that the property $P(A) :=$"A is Noetherian" is an affine-local property.

2.2 Irreducible and integral schemes

Definition 2.2.1. Let X be a scheme.

(1) X is called *connected* (resp. *irreducible*) if its underlying topological spaces is connected (resp. irreducible) in the sense of Exercise 1.5.8 and Definition 1.1.16.

(2) X is called *reduced* if for any open $U \subset X$, the ring $\mathcal{O}_X(U)$ is reduced, i.e., has no non-zero nilpotent elements.

(3) X is called *integral* if it is reduced and irreducible.

Lemma 2.2.2. A scheme X is integral iff $\mathcal{O}_X(U)$ is an integral domain for all open $U \subset X$.

Proof. Let X be integral. By definition and by Exercise 1.1.20, open subschemes of X are again integral. So it is enough to prove $A := \mathcal{O}_X(X)$ is a domain. Suppose that $f, g \in A$ satisfy $fg = 0$. Then $X = V(f) \cup V(g)$, so $X = V(f)$ say (by irreducibility). We claim that $f = 0$. To check this, we may replace X by an affine open subscheme and assume X is affine. Then $X = \operatorname{Spec} A = V(f)$ means $f^n = 0$ for some $n \gg 0$ (Exercise 1.1.22(4)), so that $f = 0$ since X is reduced.

Conversely, we use Exercise 1.1.20 to show X is irreducible. Let $U, V \subset X$ be open. If they do not intersect then

$$\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$$

by the sheaf property, but if this ring is a domain then one of the factors must be 0, i.e., U or V must be empty. $\square$

Exercises

Exercise 2.2.3. Prove that a scheme X is reduced iff all the stalks $\mathcal{O}_{X,x}$ are reduced (local) rings.

Prove that if X is integral, then all the stalks $\mathcal{O}_{X,x}$ are domains. Show by (a quite primitive) example that the converse fails. In other words, being integral is not a local but a global property of a scheme.

Exercise 2.2.4. Suppose X is an integral scheme. Show that for any inclusion of open subsets $\varnothing \neq V \subset U (\subset X)$ the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is injective. Deduce that the natural maps $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ (for $x \in U$) are injective.

Exercise 2.2.5. Consider the natural map $\mathbf{Q}[x] \subset \mathbf{Q}[x, y]/y^2 - x$, and let $f : X := \operatorname{Spec} \mathbf{Q}[x, y]/y^2 - x \to Y := \mathbf{A}^1_{\mathbf{Q}} = \operatorname{Spec} \mathbf{Q}[x]$ be the induced map of affine schemes. For each of the following points $\mathfrak{p} \in Y$, describe $f^{-1}(y)$, i.e., describe whether it is connected and whether it is reduced:

- $\mathfrak{p} = (x - 1)$
- $\mathfrak{p} = (x + 1)$
- $\mathfrak{p} = (x)$
- $\mathfrak{p} = (0)$.

2.3 The Proj construction

Recall that the n-dimensional *complex projective space* is defined as

$$\mathbf{CP}^n \quad := \quad \mathbf{C}^{n+1} \backslash \{0\} / (x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \text{ for all } \lambda \in \mathbf{C}^\times \tag{2.3.1}$$

It is known that this is a compact complex n-dimensional manifold. For example, $\mathbf{CP}^1$ is homeomorphic to the 2-sphere. Our goal in this section is to construct a scheme denoted $\mathbf{P}^n$ whose $\mathbf{C}$-points can be identified with $\mathbf{CP}^n$. The above definition suggests to consider a quotient

$$\mathbf{A}^{n+1}\backslash\{0\}/\mathbf{G}_{\mathrm{m}},$$

where the multiplicative group $\mathbf{G}_{\mathrm{m}}$ (Exercise 1.6.33) acts on $\mathbf{A}^{n+1}\backslash\{0\} \subset \mathbf{A}^{n+1}$ by multiplication.

The handling of quotients of schemes by actions of group schemes is in general much more subtle than, say, the quotient of a topological space by the action of a topological group. The $\mathbf{G}_{\mathrm{m}}$-action on $\mathbf{A}^n\backslash\{0\}$ is a free action, and in this case one can confirm the existence of the above quotient by hand; cf. Exercise 2.3.19 for further an explanation of the appearance of the gradings below.

Definition 2.3.2. (1) A *graded ring* (it would be more precise to call it an $\mathbf{N}$-graded ring) is a ring of the form $A = \bigoplus_{n \geqslant 0} A_n$ satisfying the condition that the multiplication satisfies $A_m \cdot A_n \subset A_{m+n}$.

(2) A *graded ring homomorphism* $f : A \to B$ is a ring homomorphism such that $f(A_n) \subset B_n$ for all n.

(3) An element $a \in A$ is called *homogeneous of degree* n if $a \in A_n$. In this event we write $\deg a = n$. It is called *homogeneous* if it is homogeneous of degree n for some n. (Note this implies that $0 \in A$ is considered to be homogeneous of any degree.)

(4) A *graded ideal* $I \subset A$ is an ideal such that I is generated by its homogeneous elements.

(5) The *irrelevant ideal* is

$$A_+ := \bigoplus_{n>0} A_n.$$

(It is a graded ideal in A.)

For a graded ring $A = \bigoplus_n A_n$, we note that $A_0 \subset A$ is a subring. Also, A_+ is a graded ideal. If, above, we replace "$n \geqslant 0$" by "$n \in \mathbf{Z}$", we obtain the notion of a $\mathbf{Z}$-graded ring.

Example 2.3.3. For a ring B (not equipped with a grading), the polynomial ring $A = B[t_0, \dots, t_n]$ is a graded ring if we declare A_n to consist of *homogeneous polynomials* of degree n. The irrelevant ideal is $A_+ = (t_0, \dots, t_n)$. In other words, $\operatorname{Spec} A = \mathbf{A}^{n+1}_B$ and $V(A_+)$ is the origin in this $n+1$-dimensional affine space (over $\operatorname{Spec} B$).

Definition 2.3.4. The *Proj construction* is

$$\operatorname{Proj} A := \{\mathfrak{p} \in \operatorname{Spec} A\backslash V(A_+) \mid \mathfrak{p} \text{ is graded}\}.$$

We endow it with the topology coming from the Zariski topology of $\operatorname{Spec} A$.

For a ring B (not carrying a grading), we define (using the grading discussed in Example 2.3.3) the *projective space* (of dimension n, over $\operatorname{Spec} B$):

$$\mathbf{P}^n_B := \operatorname{Proj} B[t_0, \dots, t_n].$$

We now elucidate the topology of $\operatorname{Proj} A$.

Lemma 2.3.5. A basis of the topology on $\operatorname{Proj} A$ is given by the subsets

$$D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} A, f \notin \mathfrak{p}\} = D(f) \cap \operatorname{Proj} A.$$

where f is an arbitrary homogeneous element such that $f \in A_+$.

Proof. A priori, we have to consider $D(f) \cap \operatorname{Proj} A$ for any element $f \in A$. However, if $f = \sum_{d \geqslant 0} f_d$ is a decomposition into its homogeneous components, then a graded prime ideal $\mathfrak{p} \subset A$ satisfies $f \notin \mathfrak{p}$ iff $f_d \notin \mathfrak{p}$ for some d. Thus, it is enough to consider f homogeneous. We show it suffices to consider f homogeneous and of positive degree. Given a $\mathfrak{p} \in D_+(f)$, we will show there is a homogeneous $g \in A_+$ such that $\mathfrak{p} \in D_+(g) \subset D_+(f)$. Since $\mathfrak{p}$ does (by definition of Proj) not contain A_+, there is some $h \in A_+$, $h \notin \mathfrak{p}$. Then $f \notin \mathfrak{p}$ implies $fh \notin \mathfrak{p}$, and fh is homogeneous, $\deg(fh) = \deg f + \deg h > 0$. So, we can put $g = fh$ above. □

For $f \in A_+$ homogeneous, we consider the localization $A[f^{-1}]$, which has a $\mathbf{Z}$-grading given by

$$\deg(\frac{a}{f^n}) = \deg a - n \deg f.$$

We let $A[f^{-1}]_0$ be the degree 0 part of this ring, i.e.,

$$A[f^{-1}]_0 = \{\frac{a}{f^n} | \deg a = n \deg f\}.$$

Proposition 2.3.6. For f homogeneous of positive degree, the ring homomorphisms

$$A \to A[f^{-1}] \supset A[f^{-1}]_0$$

induce homeomorphisms

$$D_+(f) \overset{\cong}{\leftarrow} \{\mathbf{Z}\text{-graded prime ideals in } A[f^{-1}]\} \overset{\cong}{\to} \operatorname{Spec}(A[f^{-1}]_0).$$

Proof. We abbreviate $S := A[f^{-1}] \supset S_0 := A[f^{-1}]_0$. By definition, $D_+(f) = D(f) \cap \operatorname{Proj} A$ is a subspace (with the induced topology) of $D(f) = \operatorname{Spec} S$. We note that $\mathbf{Z}$-graded ideals and $\mathbf{N}$-graded ideals in S agree, which proves the left hand bijection.

Let us write φ for the right hand map. As usual, it is given by taking preimages under the inclusion $S_0 \subset S$, i.e., $\mathfrak{p} \mapsto \varphi(\mathfrak{p}) = \mathfrak{p} \cap S_0$.

We show that φ is injective. Indeed, for two $\mathbf{Z}$-graded prime ideals, $\mathfrak{p}$ and $\mathfrak{q}$ not containing f, we have $\varphi(\mathfrak{p}) \subset \varphi(\mathfrak{q})$ iff $\mathfrak{p} \subset \mathfrak{q}$. Clearly, we have "$\Leftarrow$". Conversely, given a homogeneous element $a \in \mathfrak{p}$, we will show $a \in \mathfrak{q}$. Let $n := \deg a \geqslant 0$, $d := \deg f > 0$. Then $\frac{a^d}{f^n} \in \mathfrak{p}S \cap S_0 = \varphi(\mathfrak{p}) \subset \varphi(\mathfrak{q})$, so there is some homogeneous $x \in \mathfrak{q}$ such that $\frac{a^d}{f^n} = \frac{x}{f^m} (\in S)$, where $md = \deg x$. Therefore, for $e \gg 0$, we have $f^e(f^m a^d - f^n x) = 0 \in \mathfrak{q} \subset A$, so that $f^m a^d - f^n x \in \mathfrak{q}$, and therefore, since $x \in \mathfrak{q}$, $f^m a^d \in \mathfrak{q}$, and again using that $f \notin \mathfrak{q}$ and $d = \deg f > 0$, we see $a \in \mathfrak{q}$.

We show that φ is surjective. Given a prime ideal $\mathfrak{p}_0 \subset S_0$, we define a subset (already suggestively denoted) $\mathfrak{p} := \bigoplus_{n \geqslant 0} M_n$, where $M_n := \{a \in S_n, \frac{a^d}{f^n} \in \mathfrak{p}_0\}$. We have $\mathfrak{p} \cap S_0 = \mathfrak{p}_0$, so $\mathfrak{p}$ will be a preimage under φ once we show it is a graded prime ideal. We have $a \in M_n$ iff $a^2 \in M_{2n}$ ("$\Rightarrow$" is clear; "$\Leftarrow$": if $\frac{a^{2d}}{f^{2n}} = (\frac{a^d}{f^n})^2 \in \mathfrak{p}_0$, then $\frac{a^d}{f^n} \in \mathfrak{p}_0$.) For $a, b \in M_n$ we have $a^2 + 2ab + b^2 \in M_{2n}$. Indeed, this follows from expanding $(a^2 + 2ab + b^2)^d$ into monomials and using identities such as $\frac{(ab)^d}{f^{2n}} = \frac{a^d}{f^n} \frac{b^d}{f^n} \in \mathfrak{p}_0$. Therefore, $a + b \in M_n$, so $\mathfrak{p}$ is an ideal. It is a graded ideal since by definition it is generated by homogeneous elements. Finally, it is a prime ideal. It suffices to check that for two homogeneous elements $a \in S_n$, $b \in S_m$ with $ab \in \mathfrak{p}$ we have $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. From $ab \in \mathfrak{p} \cap S_{n+m} = M_{n+m}$, so again $\mathfrak{p}_0 \ni \frac{(ab)^d}{f^{n+m}} = \frac{a^d}{f^n} \frac{b^d}{f^m}$ and the primality of $\mathfrak{p}_0$ we obtain our claim. We have proved that φ is a bijection.

To see that both maps are in fact homeomorphisms it suffices to use Lemma 2.3.5 according to which a basis for the topology on $D_+(f)$ is given by $D_+(f) \cap D_+(g) = D_+(fg)$, where g is a homogeneous element of positive degree. Under the above bijections, this corresponds to $\operatorname{Spec}(A[(fg)^{-1}]_0)$. We claim that this identifies with the basic open subset $D(g)$ inside $\operatorname{Spec} A[f^{-1}]_0$. Indeed, this follows from the following equality (which is readily confirmed, note both are subrings of $A[(fg)^{-1}]$):

$$(A[f^{-1}]_0)[(\frac{g^{\deg f}}{f^{\deg g}})^{-1}] = A[(fg)^{-1}]_0. \tag{2.3.7}$$

Definition 2.3.8. We let $\operatorname{Proj} A$ be the scheme that is obtained by glueing (in the sense of Lemma 2.1.5) the $\operatorname{Spec}(A[f^{-1}]_0)$, where $f \in A_+$ is an arbitrary homogeneous element.

Example 2.3.9. We continue exploring $\mathbf{P}^n = \operatorname{Proj} \mathbf{Z}[t_0, \dots, t_n]$. The irrelevant ideal A_+ is given by $A_+ = (t_0, \dots, t_n)$, so that $\mathbf{P}^n$ is covered by $D_+(t_i)$, $0 \leqslant i \leqslant n$. We observe that there is a ring isomorphism

$$\mathbf{Z}[u_0, \dots, \widehat{u_i}, \dots, u_n] \xrightarrow{\cong} \mathbf{Z}[t_0, \dots, t_n, t_i^{-1}]_0, u_j \mapsto \frac{t_j}{t_i}$$

(as usual, the notation $\widehat{u_i}$ means that u_i is missing). According to the above definition of the scheme structure, we then have isomorphisms of schemes

$$\begin{tikzcd}
D_+(t_i t_j) = \operatorname{Spec} \mathbf{Z}[t_0, \dots, t_n, (t_i t_j)^{-1}]_0 \arrow[r, "\cong"] \arrow[d, hook] & \operatorname{Spec} \mathbf{Z}[u_0, \dots, \widehat{u_i}, \dots, u_n, u_j^{-1}] \arrow[d, hook] \\
D_+(t_i) = \operatorname{Spec} \mathbf{Z}[t_0, \dots, t_n, t_i^{-1}]_0 \arrow[r, "\cong"] & \operatorname{Spec} \mathbf{Z}[u_0, \dots, \widehat{u_i}, \dots, u_n] = \mathbf{A}^n.
\end{tikzcd} \tag{2.3.10}$$

Here, the top right subspace is the open subspace $D(u_j) \subset \mathbf{A}^n$. In other words, $\mathbf{P}^n$ is covered by $n+1$ open affine subschemes U_i that are each isomorphic to $\mathbf{A}^n$. Their pairwise intersections $U_i \cap U_j$ (for $i \neq j$) are isomorphic to $\mathbf{G}_{\mathrm{m}} \times \mathbf{A}^{n-1}$.

This shows that the construction of $\mathbf{P}^n$ recovers the topological structure of $\mathbf{CP}^n$, which is covered by the subsets of cosets of elements of the form $(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ (for $i = 0, \dots, n$). In complex analysis, *Liouville's theorem* asserts that any holomorphic function $f : \mathbf{CP}^n \to \mathbf{C}$ is constant. The following statement is the algebro-geometric incarnation thereof.

Lemma 2.3.11. Let B be a ring and $A = B[t_0, \dots, t_n]$ be equipped with its standard grading, as before, and put $X := \operatorname{Proj} A = \mathbf{P}^n_B$. There is a ring isomorphism

$$B \cong \mathcal{O}_X(X).$$

Proof. We compute the ring of global sections using the sheaf condition for the covering $\mathbf{P}^n_B = \bigcup_{i=0}^n D_+(t_i)$.

$$\mathcal{O}_X(X) = \operatorname{eq}\left(\prod_i \mathcal{O}_X(D_+(t_i)) \rightrightarrows \prod_{i,j} \mathcal{O}_X(D_+(t_i t_j))\right).$$

Thus we consider $f_i \in A[t_i^{-1}]_0$ (for each $i \leqslant n$) such that $f_i = f_j \in A[(t_i t_j)^{-1}]_0$. This latter condition enforces that $f_i \in A_0$ (as opposed to the localization). ☐

The following statement recovers the set-theoretical description of $\mathbf{CP}^n$ alluded to above. The statement below does generally not hold for non-local rings; this will be adressed by the introduction of line bundles in **??** below.

Lemma 2.3.12. For any local ring A, we have the following description of the A-points

$$\begin{aligned}
\mathbf{P}^n(A) := \operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec} A, \mathbf{P}^n) &= \{(a_0, \dots, a_n) \in A^{n+1}, a_i \in A^\times \text{ for some } i\}/\sim && (2.3.13)\\
&= \{A^{n+1} \twoheadrightarrow A\} && (2.3.14)
\end{aligned}$$

with $\sim$ being defined as in (2.3.1) and in the second line we have the set of surjections of A-modules as indicated.

Proof. We use Exercise 2.0.4: any map $\operatorname{Spec} A \to \mathbf{P}^n$ factors through $D_+(t_i)$ for $0 \leqslant i \leqslant n$.

A morphism of (affine) schemes

$$\operatorname{Spec} A \to D_+(t_i) = \operatorname{Spec} \mathbf{Z}[t_0, \dots, t_n, t_i^{-1}]_0 = \operatorname{Spec} \mathbf{Z}[u_0, \dots, \widehat{u_i}, \dots, u_n]$$

is a collection of elements $a_\bullet := (a_0, \dots, \widehat{a_i}, \dots, a_n \in A)$; in fact a_k is the image of $\frac{t_k}{t_i}$ (for $k \neq i$). Such a morphism will be identified with a morphism $\operatorname{Spec} A \to D_+(t_j)$ iff it factors through their intersection, $D_+(t_i t_j)$, as shown below. In view of the above discussion, we denote the above element u_k by $\frac{t_k}{t_i}$ at the bottom right below, and likewise for the bottom left:

$$\begin{tikzcd}
 & \operatorname{Spec} A \arrow[dl, "a_\bullet"'] \arrow[d] \arrow[dr, "b_\bullet"] & \\
D_+(t_i) \arrow[d, "\cong"] & D_+(t_i t_j) \arrow[l] \arrow[r] & D_+(t_j) \arrow[d, "\cong"] \\
\operatorname{Spec} \mathbf{Z}[\frac{t_n}{t_i}, n \neq i] & & \operatorname{Spec} \mathbf{Z}[\frac{t_n}{t_j}, n \neq j]
\end{tikzcd}$$

If we denote the collection of elements corresponding to the map $\operatorname{Spec} A \to D_+(t_j)$ by $b_0, \dots, \widehat{b_j}, \dots, b_n \in A$, with $b_i \in A^\times$ this means that a_i and b_j have to be invertible, and for the remaining indices we have $a_k = b_k \frac{a_i}{b_j}$. One then confirms that this induces an identification of $\mathbf{P}^n(A)$ with the set as stated above, by sending $a_\bullet$ to $(a_0, \dots, 1, \dots, a_n)$, where 1 is in the i-th spot.

The description in (2.3.14) is equivalent to the one above, since $a_0, \dots, a_n$ in the local ring A generate A iff one of the a_i is a unit. □

The Proj construction is also the basis of the following fundamental construction.

Definition 2.3.15. Let $X = \operatorname{Spec} B$ be affine and $Z = \operatorname{Spec} B/I$ a closed subscheme. The *blow-up* of Z in X is defined as

$$\mathrm{Bl}_Z X := \operatorname{Proj} \bigoplus_{n \geqslant 0} I^n = \operatorname{Proj}(B \oplus I \oplus I^2 \oplus \dots).$$

Example 2.3.16. Consider $B = \mathbf{Z}[t_1, \dots, t_n]$, so $X = \mathbf{A}^n$, and $I = (t_1, \dots, t_n)$, so Z is the origin in $\mathbf{A}^n$. Let us write $A = \bigoplus_{n \geqslant 0} I^n$. In order to understand $\mathrm{Bl}_0 \mathbf{A}^n = \operatorname{Proj} A$, we use the surjection $B[x_1, \dots, x_n] \to A, x_i \mapsto t_i$. One checks that its kernel is generated by $t_i x_j - t_j x_i$ for $1 \leqslant i, j \leqslant n$. In other words, $\mathrm{Bl}_0 \mathbf{A}^n$ is the closed subscheme of $\mathbf{P}^{n-1} \times \mathbf{A}^n (= \operatorname{Proj} \mathbf{Z}[x_i] \times \operatorname{Spec} B)$ defined by the homogeneous equations $t_i x_j - t_j x_i$. We analyze the fibers of the map $\varphi : \mathrm{Bl}_0 \mathbf{A}^n \to \mathbf{A}^n$:

- The fiber $\varphi^{-1}(0)$ is $\operatorname{Proj} A/(t_i) = \operatorname{Proj} \mathbf{Z}[x_j] = \mathbf{P}^{n-1}$.
- The restriction to the complement of the origin, $U := \mathbf{A}^n \backslash \{0\}$, is an isomorphism. Indeed, it is enough to check that the fiber on each $D(t_i)$ is an isomorphism, but

$$\operatorname{Proj} \mathbf{Z}[x_1, \dots, t_1, \dots, t_i^{-1}]/t_i x_j - t_j x_i = \operatorname{Spec} \mathbf{Z}[t_1, \dots, t_i^{-1}]$$

  since $x_j = t_i^{-1} t_j x_i$, so the preceding Proj-scheme is isomorphic to $\operatorname{Proj} \mathbf{Z}[x_1, t_1, \dots, t_n, t_i^{-1}]$ (with x_1 in graded degree 1).

In other words, we have a diagram whose two squares are cartesian:

$$\begin{tikzcd}
\mathbf{P}^{n-1} \arrow[r] \arrow[d] & \mathrm{Bl}_0 \mathbf{A}^n \arrow[d, "\varphi"] & \mathbf{A}^n \backslash \{0\} \arrow[l] \arrow[d, equal] \\
0 \arrow[r] & \mathbf{A}^n & \mathbf{A}^n \backslash \{0\} \arrow[l]
\end{tikzcd} \tag{2.3.17}$$

Exercises

Exercise 2.3.18. Prove that $\operatorname{Proj} \mathbf{Z}[t_1, t_2, \dots]$ (infinitely many variables) is not quasi-compact. (This is in contrast with Lemma 1.1.10, which shows that affine schemes are always quasi-compact.)

Exercise 2.3.19. Recall from Exercise 1.6.33 that $\mathbf{G}_{\mathrm{m}} = \operatorname{Spec} \mathbf{Z}[t^{\pm 1}]$ is the multiplicative group whose multiplication $\mu : \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$, inverse $\iota : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$, and neutral element $1 : \operatorname{Spec} \mathbf{Z} \to \mathbf{G}_{\mathrm{m}}$ are obtained by applying Spec to

$$\begin{aligned} \mathbf{Z}[t^{\pm 1}] &\to \mathbf{Z}[u^{\pm 1}] \otimes \mathbf{Z}[v^{\pm 1}] = \mathbf{Z}[u^{\pm 1}, v^{\pm 1}], t \mapsto u \otimes v, \\ \mathbf{Z}[t^{\pm 1}] &\to \mathbf{Z}[t^{\pm 1}], t \mapsto t^{-1}, \\ \mathbf{Z} &\to \mathbf{Z}[t^{\pm 1}]. \end{aligned}$$

Let $X = \operatorname{Spec} A$ be an affine scheme. Similarly to the case of a group acting on a set, we say that a $\mathbf{G}_{\mathrm{m}}$-*action* on X is a map (of affine schemes)

$$\mathrm{act} : \mathbf{G}_{\mathrm{m}} \times X \to X$$

satisfying the usual axioms of a group action such as the commutativity of the diagram

$$\begin{tikzcd} \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} \times X \arrow[r, "\mathrm{id}_{\mathbf{G}_{\mathrm{m}}} \times \mathrm{act}"] \arrow[d, "\mu \times \mathrm{id}_X"] & \mathbf{G}_{\mathrm{m}} \times X \arrow[d, "\mathrm{act}"] \\ \mathbf{G}_{\mathrm{m}} \times X \arrow[r, "\mathrm{act}"] & X. \end{tikzcd}$$

Prove that $\mathbf{G}_{\mathrm{m}}$-action on X is equivalently a $\mathbf{Z}$-grading of A, i.e. $A = \bigoplus_{n \in \mathbf{Z}} A_n$.

Hint: applying $\mathcal{O}$ to the above action map gives a map $A \to A \otimes \mathbf{Z}[t^{\pm 1}] = \bigoplus_{n \in \mathbf{Z}} A$. What does the commutativity of the diagram mean in terms of this map?

Exercise 2.3.20. Let $A = B[t_0, \dots, t_n]$ be as in Example 2.3.3.

(1) Check that the $\mathbf{G}_{\mathrm{m}}$-action on $\operatorname{Spec} A = \mathbf{A}^{n+1}_B$ given by scaling restricts to a $\mathbf{G}_{\mathrm{m}}$-action on $\operatorname{Spec} A \backslash V(A_+)$.

(2) Let T be a scheme. We equip it with the trivial $\mathbf{G}_{\mathrm{m}}$-action. For any $\mathbf{G}_{\mathrm{m}}$-equivariant map f prove that there is a unique scheme homomorphism like so:

$$\begin{tikzcd} & \operatorname{Proj} A \arrow[d, dotted] \\ \operatorname{Spec} A \backslash V(A_+) \arrow[ur] \arrow[r, "f"] & T \end{tikzcd}$$

This confirms that $\operatorname{Proj} A$ is (in the category of schemes) a quotient of $(\operatorname{Spec} A \backslash V(A_+))/\mathbf{G}_{\mathrm{m}}$.

2.4 Fiber products

Lemma 2.4.1. The category Sch has a final object, namely $\operatorname{Spec} \mathbf{Z}$. That is, for any scheme X there is exactly one map $X \to \operatorname{Spec} \mathbf{Z}$.

Proof. By Theorem 1.6.14, we have $\operatorname{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} \mathbf{Z}) = \operatorname{Hom}_{\mathrm{Rings}}(\mathbf{Z}, \mathcal{O}_X(X))$, and for any ring R, there is exactly one ring homomorphism $\mathbf{Z} \to R$. □

Proposition 2.4.2. The category Sch of schemes has fiber products. That is, for any two maps $X' \to X$ and $Y \to X$ there is a scheme Y' with the following universal property: for any scheme T and maps $T \to X'$ and $T \to Y$ making the outer diagram commute there is a unique map $T \to Y'$ making the remainder of the diagram commute:

$$\begin{tikzcd} T \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Y' \arrow[r] \arrow[d] & X' \arrow[d] \\ & Y \arrow[r] & X. \end{tikzcd}$$

Proof. We only prove this in the case where $X' = \operatorname{Spec} A'$, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. We claim that in this case the affine scheme $Y' = \operatorname{Spec}(B \otimes_A A')$ is a fiber product. Indeed, the universal property as above does hold if $T = \operatorname{Spec} R$ is also an affine scheme: passing to the opposite category of rings, cf. (1.6.19), this is just the assertion that $B \otimes_A A'$ is the pushout of the diagram

$$\begin{tikzcd}
R & & \\
& B \otimes_A A' \arrow[ul, dotted] & A' \arrow[l] \arrow[ull, "r"'] \\
& B \arrow[u] \arrow[uul, "s"] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

This holds by the very definition of the tensor product: the diagonal dotted arrow is the unique map sending $b \otimes a'$ to $s(b) \cdot r(a')$. If T is any scheme, then we have $\operatorname{Hom}_{\mathrm{Sch}}(T, \operatorname{Spec} E) = \operatorname{Hom}_{\mathrm{AffSch}}(\operatorname{Spec} \mathcal{O}_T(T), \operatorname{Spec} E)$ by Theorem 1.6.14, i.e., we may replace T by $\operatorname{Spec} \mathcal{O}_T(T)$ and reduce to the case of T being affine.

The case of not necessarily affine X, X', Y is reduced in several steps to the affine case. See, e.g., [Stacks, Tag 01JM] and [Stacks, Tag 01JS]. □

Example 2.4.3. For a ring A, we have

$$\mathbf{A}^n_A = \mathbf{A}^n_{\mathbf{Z}} \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} A,$$

$$\mathbf{P}^n_A = \mathbf{P}^n_{\mathbf{Z}} \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} A,$$

where $\operatorname{Spec} A \to \operatorname{Spec} \mathbf{Z}$ is the unique map (corresponding to the unique ring homomorphism $\mathbf{Z} \to A$).

Warning 2.4.4. The underlying set of a (fiber) product of schemes $X \times_Y Z$ is *not* in general the (fiber) product of the underlying sets. This issue already manifests itself for affine schemes. Here are two concrete examples:

(1) For an algebraically closed field $\overline{k}$, consider $\mathbf{A}^2_{\overline{k}} = \mathbf{A}^1_{\overline{k}} \times_{\operatorname{Spec} \overline{k}} \mathbf{A}^1_{\overline{k}}$. In $\mathbf{A}^2_{\overline{k}} = \operatorname{Spec} \overline{k}[t_1, t_2]$, we have closed subsets such as $\Delta := V(t_1 - t_2)$, i.e., the diagonal. Its generic point is not of the form $\mathbf{A}^1_{\overline{k}} \times \{x\}$ (for some point $x \in \mathbf{A}^1_{\overline{k}}$).

(2) If k is a field and k'/k is a field extension then $X := \operatorname{Spec} k' \times_{\operatorname{Spec} k} \operatorname{Spec} k' = \operatorname{Spec}(k' \otimes_k k')$ is rarely consisting of a single point. If, say, the extension is a finite Galois extension, generated by an element $x' \in k'$ with minimal polynomial $p(t) \in k[t]$, then

$$k' \otimes_k k' = k[t]/p(t) \otimes_k k' = k'[t]/p(t) = \prod_{i=1}^{\deg p} k',$$

where the right hand isomorphism is using the splitting of p in k' into linear factors. Thus X consists of $\deg p$ points in this case.

If $k' = k(t)$, then $\dim X = 1$ (Exercise 1.9.7).

(3) On the positive side, though, if k is an algebraically closed field and $X_1, X_2 / \operatorname{Spec} k$ are schemes of finite type (Definition 2.7.2), then we have

$$(X_1 \times_{\operatorname{Spec} k} X_2)^{\mathrm{cl}} = X_1^{\mathrm{cl}} \times X_2^{\mathrm{cl}},$$

where the superscript cl denotes the subset of closed points. Indeed, this assertion reduces to the case where X_i are affine, which is discussed in Exercise 1.9.5.

Corollary 2.4.5. The category Sch has finite products: $X \times Y = X \times_{\operatorname{Spec} \mathbf{Z}} Y$.

Definition 2.4.6. Let P be a property of morphisms of schemes. We say "P is *stable under pullback*" if for any pullback diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{2.4.7}$$

we have the implication

$$P(f) \Rightarrow P(f').$$

This turns out to be a very important organizational principle. We will meet many more (and more meaningful) examples of this soon.

Example 2.4.8. The condition "f is surjective" is stable under pullback. Indeed, a point in $y' \in Y'$ in the diagram 2.4.7 yields a map $\operatorname{Spec} k \to Y'$ (with k being the residue field of y'). By the surjectivity, there is some $x \in X$ such that $f(x) = g(y')$. Using Exercise 2.0.3 we see that there is a field extension $k \supset k(y')$ and a map $\operatorname{Spec} k \to X$ as indicated below, whose image is x, making the outer part of the diagram commutative:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[dd] \arrow[dr, dotted] \arrow[drr] & & \\ & X \times_Y Y' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y') \arrow[r] & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Since the right hand square is cartesian, there is a unique dotted map making everything commutative; in particular this produces a point in $X \times_Y Y'$ that maps to y'.

Non-example 2.4.9.

- The condition "f is injective" is *not* stable under pullback, as the example in Warning 2.4.4(2) shows.
- The condition "f has finite fibers" is not stable under pullback: one can show that for a field k and an algebraic closure $\overline{k}$, $\operatorname{Spec}(\overline{k} \otimes_k \overline{k})$ is homeomorphic to the (absolute) Galois group $\operatorname{Gal}(\overline{k}/k)$ (which is usually infinite); but of course $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ is a bijection (of a singleton). A remedy for this is discussed in Exercise 2.8.4.

2.5 Affine morphisms

Definition 2.5.1. A morphism $f : X \to Y$ of schemes is called *affine* if there is an open covering of Y by open affines $V \subset Y$ for which the preimages $f^{-1}(V) := V \times_Y X$ are again affine schemes.

For example, a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ between affine schemes is affine. By contrast, for $n \geqslant 1$, the structural map $\mathbf{P}^n \to \operatorname{Spec} \mathbf{Z}$ is not affine: its pullback to $V = B$, $B = \mathbf{Z}[1/n]$ is given by $\mathbf{P}^n_B$, but the natural map

$$\mathbf{P}^n_B \to \operatorname{Spec} \mathcal{O}_{\mathbf{P}^n_B}(\mathbf{P}^n_B) = \operatorname{Spec} B$$

(cf. Lemma 2.3.11) is not an isomorphism.

Lemma 2.5.2. A morphism $f : X \to Y$ is affine iff for *any* open affine $V \subset Y$, $f^{-1}(V)$ is affine.

One can prove this using the affine communication lemma (Proposition 2.1.7) and Exercise 2.5.3, see [Vak17, Proposition 7.3.4] (or [Har83, Exercise II.2.17] for a slightly different, but essentially equivalent approach).

Exercises

Exercise 2.5.3. Let X be a quasi-compact and *quasi-separated* scheme. The latter means that for any two open affine subschemes $U = \operatorname{Spec} A, V = \operatorname{Spec} B \subset X$, their intersection $U \cap V$ is quasi-compact.

Let $f : X \to \mathbf{A}^1$ be a morphism of schemes (equivalently, by Theorem 1.6.14, an element $f \in \mathcal{O}_X(X)$). Denote by $X_f := \mathbf{G}_\mathrm{m} \times_{\mathbf{A}^1} X$ (where the pullback is formed using f). Prove that there is an isomorphism

$$\mathcal{O}_X(X)[f^{-1}] \xrightarrow{\cong} \mathcal{O}_{X_f}(X_f).$$

2.6 Open and closed embeddings

Let $f : X \to Y$ be a map of schemes.

Definition 2.6.1.
- f is called a *locally closed embedding* iff the following two conditions hold:
  (1) The underlying map of topological spaces of f is a homeomorphism $X \cong f(X)$ and, again on the level of the underlying spaces, $f(X)$ is open in its closure $\overline{f(X)}$.
  (2) For each $x \in X$ and $y := f(x)$, the induced map on stalks
  $$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \tag{2.6.2}$$
  is surjective.
- f is called an *open embedding* if it satisfies these two conditions and $f(X)$ is open and (2.6.2) is an isomorphism.
- f is called a *closed embedding* if it satisfies these two conditions and $f(X)$ is closed.

If X is homeomorphic to an open subset $f(X) \subset Y$, then the maps (2.6.2) are isomorphisms if and only if $\mathcal{O}_Y|_{f(X)} \to \mathcal{O}_X$ is an isomorphism. This follows from Exercise 1.6.28.

Lemma 2.6.3. For a map $f : X \to Y$, the following are equivalent:
(1) f is a closed immersion,
(2) f is affine, and if for any $\operatorname{Spec} B \subset Y$, the preimage $f^{-1}(\operatorname{Spec} B)$, which is of the form $\operatorname{Spec} A$ (since f is affine) is such that the induced map $B \to A$ is surjective.

Proof. The main idea for (1) ⇒ (2) is this: if $X \to Y$ is a closed immersion, then so is $X \cap U \to U$, for any open $U \subset Y$. Indeed, we have $\mathcal{O}_{Y,y} = \mathcal{O}_{U,y}$ for $y \in U$ etc. So we may assume $Y = \operatorname{Spec} B$ is affine. If we put $A := \mathcal{O}_X(X)$, we get a canonical map $\gamma : X \to \operatorname{Spec} A$ (adjoint to the identity of $\mathcal{O}_X(X)$ under the adjunction established in Theorem 1.6.14). More precisely, there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\gamma"] \arrow[dr, "f"'] & & \operatorname{Spec} A \arrow[dl, "\delta"] \\ & Y = \operatorname{Spec} B. & \end{tikzcd}$$

One then shows that γ is an isomorphism; cf. [Stacks, Tag 01IN] for an argument involving the concept of a quasi-coherent sheaf or alternatively [GW20, Theorem 3.42] for a slightly more involved, but completely elementary argument. □

In particular, for a closed immersion into an affine scheme $Z \to \operatorname{Spec} A$, Z is again affine (in contrast to the case of open subschemes, cf. Example 2.1.2).

Exercises

Exercise 2.6.4. Show that a surjective closed embedding $X \to Y$ need not be an isomorphism. At least if Y is affine, name an appropriate additional condition on Y that ensures that any surjective closed embedding with target Y is an isomorphism.

Exercise 2.6.5. For a scheme X and two closed subschemes $Z_1, Z_2 \subset X$, the *scheme-theoretic intersection* is defined as the fiber product

$$Z_1 \cap Z_2 := Z_1 \times_X Z_2.$$

(1) For $X = \mathbf{A}^2 = \operatorname{Spec} \mathbf{Z}[x, y]$, and $Z_1 = V(y - x^2)$, $Z_2 = V(y)$ compute $Z_1 \cap Z_2$. Is it reduced?

(2) Compute the scheme-theoretic intersection of $V(y^2 - x^2)$ and $V(y)$ in $\mathbf{A}^2$.

Exercise 2.6.6. Let $f : A \to B$ be a ring homomorphism. Prove that $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is an open immersion if and only if φ is locally an isomorphism in the sense that there are $f_i \in A$ (without necessarily $\bigcup D(f_i) = \operatorname{Spec} A$) such that $\operatorname{Spec} B[f_i^{-1}] \to \operatorname{Spec} A[f_i^{-1}]$ is an isomorphism.

2.7 Finiteness conditions

In this entire section, let $f : X \to Y$ be a morphism of schemes.

Definition and Lemma 2.7.1. The following conditions are equivalent; if they hold we call f *quasi-compact*:
(1) for any quasi-compact open $U \subset Y$, $f^{-1}(U)$ is quasi-compact,

(2) for any affine $U = \operatorname{Spec} B \subset Y$, $f^{-1}(U)$ is quasi-compact.

Proof. Any quasi-compact $U \subset Y$ admits a cover by finitely many affines $U = \bigcup_{i=1}^n \operatorname{Spec} B_i$, and $f^{-1}(U) = \bigcup_i f^{-1}(\operatorname{Spec} B_i)$. □

Therefore, a scheme X is quasi-compact iff the unique map $X \to \operatorname{Spec} \mathbf{Z}$ is quasi-compact in this sense. If X is Noetherian (Definition 2.1.10) then $f : X \to Y$ is automatically quasi-compact. Indeed, X is then a Noetherian topological space (Definition 1.1.14), and *any* open in X is quasi-compact (Exercise 1.1.18). Examples of non-quasi-compact morphism include

- $\bigsqcup_{i \in I} \operatorname{Spec} k \to \operatorname{Spec} k$, for an infinite set I,
- The inclusion of infinite-dimensional affine space $X := \mathbf{A}^\infty (:= \operatorname{Spec} \mathbf{Z}[t_1, t_2, \dots])$ into the infinite-dimensional affine space with doubled origin, i.e., Y is obtained by glueing two copies of X along the open subscheme given by the complement of the origin, cf. Exercise 1.1.18.

Definition 2.7.2. A morphism $f : X \to Y$ of schemes is called *locally of finite type* (resp. *locally of finite presentation*), if for any affine open $\operatorname{Spec} B \subset Y$ and any affine open $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B) (\subset X)$ the induced map $B \to A$ is such that A is a finitely generated B-algebra (resp. finitely presented).

We say f is *of finite type* if it is locally of finite type and quasi-compact.

Example 2.7.3. For any ring A, the structural map $\mathbf{P}^n_A \to \operatorname{Spec} A$ (obtained by glueing the maps $D_+(t_i) \to \operatorname{Spec} A$ induced by the inclusion $A \subset A[t_0, \dots, t_n, t_i^{-1}]_0$) is of finite type. Indeed, $\mathbf{P}^n_A$ is quasi-compact since it is covered by finitely many affine schemes, which are quasi-compact, and the $D_+(t_i) \cong \mathbf{A}^n_A$ are of finite type.

Definition 2.7.4. We say $f : X \to Y$ is *finite* if for any open affine $\operatorname{Spec} B \subset Y$ the preimage $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ is affine (i.e., f is affine) and the induced ring homomorphism $B \to A$ is such that A is a finite B-module.

Reiterating the comment in Remark 1.7.2, being finite is a much stronger condition than being finitely presented. By definition, a closed embedding is finite.

2.8 Permanence properties of morphisms

Definition 2.8.1. Let P be a property of a morphism of schemes. We write $P(f)$ if some morphism $f : X \to Y$ has the property P.

- We say "P is stable under composition" if $P(g)$ and $P(f)$ implies $P(g \circ f)$ (for any two composable morphisms g and f).
- We say "P is local on the target" if for any open covering $Y = \bigcup U_i$ we have

$$P(f^{-1}(U_i) \xrightarrow{f} U_i) \forall i \Rightarrow P(f).$$

  (Note that the converse holds if P is stable under base change.) Here and throughout below, $f^{-1}(U_i) := U_i \times_Y X$.

- We say "P is local on the source" if for any open covering $X = \bigcup U_i$, we have

$$P(U_i \subset X \to Y) \forall i \Rightarrow P(X \to Y).$$

Lemma 2.8.2. The following properties are stable under composition, stable under pullback and local on the target:
(1) locally closed immersion,

(2) open immersion,

(3) closed immersion,

(4) affine,

(5) quasi-compact,

(6) locally of finite type (*),

(7) of finite type,

(8) finite
The properties marked (*) are also local on the source (but the others are not).

Proof. To illustrate the technique, we discuss this for closed immersions. The stability under composition and the locality on the target is straightforward from Lemma 2.6.3. To see it is stable under base change, consider a pullback diagram with f a closed immersion

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

To show that f' is a closed immersion if f is one, we may by locality on the target assume Y' is affine. Then, by stability of affine maps under base change, X' is also affine and the map $\mathcal{O}_{Y'}(Y') \to \mathcal{O}_{X'}(X') = \mathcal{O}_{Y'}(Y') \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(X)$ is surjective since it is the tensor product of the surjection $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ with $\mathcal{O}_{Y'}(Y')$.

The map $\operatorname{Spec} \mathbf{Z} \sqcup \operatorname{Spec} \mathbf{Z} (= \operatorname{Spec}(\mathbf{Z} \times \mathbf{Z})) \to \operatorname{Spec} \mathbf{Z}$ is locally on the source an isomorphism, but not an open (or closed) immersion.

The proofs for the remaining properties are similar; see Exercise 2.8.5 for an approach to the claims for morphisms that are locally of finite type. $\square$

Exercises

Exercise 2.8.3. (Solution at p. 109) Suppose $f : X \to \operatorname{Spec} \mathbf{Z}$ is of finite type such that $X \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{F}_p$ is non-empty for infinitely many primes p. Prove that $X \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{Q}$ is also non-empty.

Exercise 2.8.4. Show that the condition "f is of finite type and f has finite fibers" is stable under pullback.

Exercise 2.8.5. (1) The following statement can be regarded as a relative version of Exercise 2.1.11: fix a ring B. For an B-algebra A, consider the property $P(A) :=$ "A is finitely generated as a B-algebra". Prove that this property is affine-local in the sense of Proposition 2.1.7.

Hint: if $b_i \in B$ are such that $\bigcup_i D(b_i) = \operatorname{Spec} B$, how can one take advantage of the faithful flatness of $B \to \prod_i B[b_i^{-1}]$?

(2) Deduce that a map $f : X \to Y$ (of schemes) is locally of finite type if for each $\operatorname{Spec} B \subset Y$, the preimage $f^{-1}(\operatorname{Spec} B)$ admits a covering by open affines $\operatorname{Spec} A_i$ with A_i being a finitely generated B-algebra.

Exercise 2.8.6. Let X, Y be schemes that are of finite type (resp. locally of finite type) over $\operatorname{Spec} A$. Prove that $X \times_{\operatorname{Spec} A} Y$ then has the same property.

Exercise 2.8.7. Taking Lemma 2.8.2(8) for granted, prove that any finite morphism f is *universally closed*, i.e., that for any pullback diagram as in (2.4.7), the pullback f' is a closed map.

2.9 Separated and proper maps

In topology, *compact Hausdorff spaces* have several enjoyable features. In this section we are going to explore the algebro-geometric analogues of this concept.

Recall that a topological space is *Hausdorff* X if for any $x \neq y \in X$ there are open neighborhoods $U \ni x$, $V \ni y$ such that $U \cap V = \varnothing$. One checks that this is equivalent to requiring the diagonal $\Delta_X = \{(x, x)\} \subset X \times X$ to be a closed subset. This motivates the next definition.

Definition 2.9.1. A morphism of schemes $f : X \to Y$ is *separated* if the diagonal map

$$\Delta : X \to X \times_Y X$$

is a closed immersion (Definition 2.6.1). A scheme X is called *separated* if the unique map $X \to \operatorname{Spec} \mathbf{Z}$ is separated.

Here, the map $\Delta := \Delta_f$ is the unique map such that the composition with the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : X \times_Y X \to X$ is the identity id_X. For any map f, Δ_f is a locally closed immersion (Exercise 2.9.18).

Example 2.9.2. Any map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is separated. Indeed, the diagonal corresponds to the multiplication map $B \otimes_A B \to B$, which is surjective, and therefore a closed immersion after passing to spectra.

Lemma 2.9.3. If X is separated, $U, V \subset X$ are affine open subschemes, then $U \cap V$ is also affine and the map

$$\mathcal{O}_X(U) \otimes_{\mathbf{Z}} \mathcal{O}_X(V) \to \mathcal{O}_V(U \cap V), f \otimes g \mapsto f|_{U \cap V} \cdot g|_{U \cap V}$$

is surjective.

Conversely, if $X = \bigcup U_i$ is a cover by open affines such that $U_i \cap U_j$ is affine for all i, j and the maps

$$\mathcal{O}_X(U_i) \otimes_{\mathbf{Z}} \mathcal{O}_X(U_j) \to \mathcal{O}_V(U_i \cap U_j) \tag{2.9.4}$$

are surjective, then X is separated.

Proof. We have a pullback diagram

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times V \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X, \end{tikzcd}$$

so if Δ is a closed embedding, so is the top horizontal map, so that $U \cap V$ is affine and the requested surjectivity holds.

Conversely, $X \times X = \bigcup_{i,j} U_i \times U_j$. By Lemma 2.8.2, Δ is a closed embedding iff its pullback $U_i \cap U_j \to U_i \times U_j$ is a closed embedding (for each i, j), but for $U_i = \operatorname{Spec} A_i$ our condition above means that

$$U_i \cap U_j = \operatorname{Spec}(A_i \otimes A_j)/I$$

for some ideal I. □

Non-example 2.9.5. Consider the scheme X obtained by glueing $U_1 = \mathbf{A}^1$ and $U_2 = \mathbf{A}^1$ along the $\mathbf{G}_\mathrm{m} \subset \mathbf{A}^1$. This scheme is called the *affine line with doubled origin*. It is not separated (which is in line with the observation that, say, $\mathbf{R} \sqcup_{\mathbf{R}\setminus\{0\}} \mathbf{R}$ is not Hausdorff): the map (2.9.4) is the multiplication

$$\mathbf{Z}[t] \otimes \mathbf{Z}[t] \to \mathbf{Z}[t^{\pm 1}],$$

which is not surjective.

Example 2.9.6. $\mathbf{P}^n$ is separated, since the covering by the affine subspaces $D_+(t_i) \cong \mathbf{A}^n$ satisfies the condition in (2.9.4). The map reads

$$\mathbf{Z}[\underline{t}, t_i^{-1}]_0 \otimes_{\mathbf{Z}} \mathbf{Z}[\underline{t}, t_j^{-1}]_0 \to \mathbf{Z}[\underline{t}, (t_i t_j)^{-1}]_0.$$

It is surjective (here $\underline{t}$ is a shorthand for $t_0, \dots, t_n$). Indeed this the right hand side identifies, as noted in (2.3.7) with

$$\mathbf{Z}[\underline{t}, t_i^{-1}]_0[(\frac{t_j}{t_i})^{-1}] = \mathbf{Z}[\underline{t}, t_j^{-1}]_0[(\frac{t_i}{t_j})^{-1}].$$

Lemma 2.9.7. Separated morphisms are stable under composition and stable under pullback. Also, the condition of being separated is local on the target.

Proof. Using the same permanence properties for closed embeddings, this can be shown by purely categorical considerations. We illustrate this for the composition: let $f : X \to Y$ and $g : Y \to Z$ be separated. Write $h := gf : X \to Z$. We can factor the diagonal Δ_h like so, where the right hand square is cartesian:

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[dr, "\Delta_h"'] & X \times_Y X = (X \times_Z X) \times_{Y \times_Z Y} Y \arrow[r] \arrow[d, "\mathrm{id} \times \Delta_g"] & Y \arrow[d, "\Delta_g"] \\ & X \times_Z X = (X \times_Z X) \times_{Y \times_Z Y} (Y \times_Z Y) \arrow[r] & Y \times_Z Y \end{tikzcd}$$

By stability of closed immersions under pullback and under composition, Δ_h is a closed embedding. □

Definition 2.9.8. A map $f : X \to Y$ is called *proper* if it is of finite type (Definition 2.7.2), separated and *universally closed* (i.e., any pullback f' of f is a closed map).

Lemma 2.9.9. Proper morphisms are stable under composition, stable under pullback. Also, properness is local on the target.

Proof. This is a consequence of the same permanence properties for separated, resp. finite type, resp. universally closed maps. □

Proposition 2.9.10. (*Valuative criterion for universally closed maps*) For a quasi-compact morphism f, the following are equivalent:
(1) f is universally closed,

(2) for any pullback $f' : X' := X \times_Y Y' \to Y'$ of f, *specializations lift* along f', i.e., i.e., if $Y' \ni y' := f(x') \rightsquigarrow y''$, then we can find $x'' \in X'$ such that $f(x'') = y''$.

(3) for any commutative square as shown below, where V is a valuation ring and $Q(V)$ its fraction field, there is a diagonal map as shown such that the two triangles are commutative:

$$\begin{tikzcd} \operatorname{Spec} Q(V) \arrow[r] \arrow[d, hook, "\eta"] & X \arrow[d, "f"] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dotted, "\exists"] & Y. \end{tikzcd} \tag{2.9.11}$$

Proof. This proof follows essentially the same pattern as its affine analogue Theorem 1.7.20.

We prove (2) ⇔ (3) (this does not use that f is quasi-compact). This was shown in the special case where X and Y are affine in (the proof of) Theorem 1.7.20, in this case we can find V such that the specialization $y \rightsquigarrow y'$ is the image of the specialization $\eta = (0) \rightsquigarrow \mathfrak{m}_V$ in $\operatorname{Spec} V$.

In general, if X and Y are not necessarily affine, we obtain (3) ⇒ (2): we pick an open affine neighborhood $\operatorname{Spec} B \subset Y$ of y' (which also contains y), and an affine neighborhood $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B) \subset X$ of x and apply the preceding case. Conversely, for (2) ⇒ (3) we can do this same reduction since the image of $\operatorname{Spec} V$ in Y is contained in an affine neighborhood of the image of the closed point (Exercise 2.0.4(1)).

(2) ⇔ (1): we have to show that if is f quasi-compact then specializations lift along f iff f is closed. Elementary arguments of point-set topology (see, e.g. [Stacks, Tag 01K9]) can be used to reduce this to the following claim (for f quasi-compact):

$$f(X) \text{ is closed } \Leftrightarrow f(X) \text{ is stable under specializations.}$$

The direction ⇒ is a simple generality of point-set topology [Stacks, Tag 0062]. The converse is exactly the content of Lemma 1.7.19 if X and Y are affine. If Y is affine and $X = \bigcup_{i=1}^n \operatorname{Spec} A_i$ (the covering is finite since f is quasi-compact!), then $f(X)$ is the image of $\operatorname{Spec}(A_1 \times \ldots \times A_n) = \bigsqcup_i \operatorname{Spec} A_i \to Y$, so this is closed by the previous case. The case of non-affine Y is reduced to the affine case again using basic point-set topology: being closed and being stable under specializations are conditions that can be checked locally on an affine open subset of Y (cf. also Exercise 1.8.10). □

Proposition 2.9.12. For a map $f : X \to Y$, the following are equivalent:
(1) f is finite,

(2) f is affine and proper.

Proof. All three conditions are local on the target, so we may assume Y is affine. In both cases this implies that X is affine, as well.

(2) ⇒ (1): if $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is universally closed, then in particular its pullback along the projection $\mathbf{A}^1_A \to \operatorname{Spec} A$ is closed, so that f is integral (Theorem 1.7.20). Since f is also of finite type, i.e., B is a finitely generated A-algebra, we see that B is actually finite We conclude by Lemma 1.7.8.

(1) ⇒ (2): any finite map is affine and therefore separated (Example 2.9.2). It is also of finite type. Finally, a finite map is integral, and therefore universally closed (this was checked in Theorem 1.7.20 for pullbacks along an affine map $\operatorname{Spec} A' \to \operatorname{Spec} A$; the case of a general pullback reduces to this since the condition of being a closed map local on the target). □

Non-example 2.9.13. The projection $\mathbf{A}^1 \to \operatorname{Spec} \mathbf{Z}$ is not proper: its pullback along $\mathbf{A}^1 \to \operatorname{Spec} \mathbf{Z}$ is the canonical projection $\mathbf{A}^2 \to \mathbf{A}^1$, which is not a closed map (Example 1.8.2).

The following theorem is the algebro-geometric analogue of the fact that $\mathbf{CP}^n$ is *compact.*

Theorem 2.9.14. The structural map $\pi : \mathbf{P}^n_A \to \operatorname{Spec} A$ is proper for any ring A.

(More generally, one can define for any scheme X, $\mathbf{P}^n_X := \mathbf{P}^n \times_{\operatorname{Spec} \mathbf{Z}} X$; then the structural map to X is proper as well.)

Proof. Since properness is stable under taking pullbacks (Lemma 2.9.9), it suffices to consider the case $X = \operatorname{Spec} \mathbf{Z}$. The map is of finite type (Example 2.7.3) and separated (Example 2.9.6). To show it is universally closed we use Proposition 2.9.10:

$$\begin{tikzcd} \operatorname{Spec} Q(V) \arrow[r] \arrow[d, hook, "\eta"'] & \mathbf{P}^n \arrow[d, "\pi"] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dotted, "\exists"] & \operatorname{Spec} \mathbf{Z}. \end{tikzcd} \tag{2.9.15}$$

We will give two independent arguments for the existence of such a lift. The *first proof* uses that the structural map π is the pullback of the map $\varphi : \operatorname{Bl}_0 \mathbf{A}^{n+1} \to \mathbf{A}^{n+1}$ (along the inclusion of the origin), cf. (2.3.17). (This argument essentially appears in [GL01, Proposition 2.1] and [Kel24, Proposition 4.3].) We prove that φ satisfies the lifting property, which implies the same property for π:

$$\begin{tikzcd} \operatorname{Spec} Q(V) \arrow[r, "d"] \arrow[d, hook, "\eta"'] & \operatorname{Bl}_0 \mathbf{A}^{n+1} \arrow[d, "\varphi"] \\ \operatorname{Spec} V \arrow[r, "a"] \arrow[ur, dotted, "\exists l"] & \mathbf{A}^{n+1}. \end{tikzcd}$$

The map a is nothing but a collection of elements $a_0, \dots, a_n \in V$. Since V is a valuation ring, one sees by induction that one of the a_i divides all the other a_j's (cf. the proof of Lemma 1.7.16). For simplicity of notation, let us say that a_0 divides $a_1, \dots, a_n$. This means that the map a factors through $\varphi \circ j$ as shown, i.e., $a = \varphi j b$:

$$\begin{tikzcd} & & \operatorname{Spec} \mathbf{Z}[t_0, \frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}] \arrow[d, hook, "j"] \\ \operatorname{Spec} Q(V) \arrow[urr, "e"] \arrow[rr, "d"] \arrow[d, hook, "\eta"'] & & \operatorname{Bl}_0 \mathbf{A}^{n+1} \arrow[d, "\varphi"] \\ \operatorname{Spec} V \arrow[rr, "a"] \arrow[uurr, "b"] \arrow[urr, "l"] & & \mathbf{A}^{n+1} = \operatorname{Spec} \mathbf{Z}[t_0, \dots, t_n]. \end{tikzcd}$$

(Concretely, b parametrizes $a_0, \frac{a_1}{a_0}, \dots$.) Putting $l := jb$ we have $\varphi \circ l = a$. To check the commutativity of the other triangle above, i.e. $d = l \circ \eta$, we note that the divisibility $a_0 | a_1$ etc. also holds in $Q(V)$, so the map d factors through j, i.e., $d = je$. Then

$$a\eta = \varphi l \eta = \varphi j b \eta = \varphi d = \varphi j e.$$

We observe that in the category of integral domains, the map

$$\iota : \mathbf{Z}[t_i] \to \mathbf{Z}[t_0, \frac{t_1}{t_0}, \dots]$$

is an epimorphism. Indeed, a map f from the target to some domain R is determined by $r_0 := f(t_0)$, $r_j := f(\frac{t_j}{t_0})$ for $j \geqslant 1$. The composition with the above inclusion ι determines r_0 and $f(t_j) = r_0 r_j$.

Thus, if we are given a second map f' to R, with $f\iota = f'\iota$, and write $r'_0 := f'(t_0)$ etc., we have $r_0 = r'_0$ and

$$r_0 r_j = r'_0 r'_j \Rightarrow r_0(r_j - r'_j) = 0 \Rightarrow r_j = r'_j$$

since R is a domain.

Since $Q(V)$ is, in particular a field, the above relation $\varphi j b \eta = \varphi j e$ implies

$$b\eta = e$$

so that

$$l\eta = jb\eta = je = d.$$

The *second proof* uses the description of the points of $\mathbf{P}^n$ in (2.3.14) (note that both V and $Q(V)$ are local rings, Lemma 1.7.16). Now, the top horizontal map amounts to giving a surjection $Q(V)^{n+1} \twoheadrightarrow Q(V)$. The composite

$$V^{n+1} \subset Q(V)^{n+1} \twoheadrightarrow Q(V)$$

has image isomorphic to V by Lemma 2.9.16. This shows the existence of a map such that the diagram (2.9.15) commutes (note that the right triangle commutes for any map since $\operatorname{Spec} \mathbf{Z}$ is a final object). □

Lemma 2.9.16. Let V be a valuation ring and M a finitely generated torsion-free V-module. Then M is free, i.e., $M \cong V^n$.

Proof. Let $V^n \twoheadrightarrow M$ be a surjection, and m_i the images of the basis vectors. If $\sum_{i=1}^n a_i m_i = 0$ for some $a_i \in V$, then there is some i such that a_i divides all the other a_j. Since M is torsion-free, we can then divide the relation by a_i and express x_i as a linear combination of the remaining x_j's. Repeating the argument with the remaining generators shows that, eventually, there is a basis of M. □

The following statement establishes a partial converse for the fact that projective spaces are proper. For a proof, see [Har83, Exercise II.4.10] or [Stacks, Tag 0200].

Proposition 2.9.17. (*Chow's lemma*) Let $f : X \to Y$ be a proper morphism, with Y being Noetherian. Then there is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "i"] & X \arrow[d, "f"] \\ \mathbf{P}^n_Y := \mathbf{P}^n \times Y \arrow[r] & Y \end{tikzcd}$$

where i is a closed immersion, and g is a proper map for which there is a non-empty open $U \subset X$ such that $g|_{g^{-1}(U)}$ is an isomorphism.

Exercises

Exercise 2.9.18. Show that for any map of schemes $f : X \to Y$, the diagonal $\Delta_f : X \to X \times_Y X$ is a locally closed embedding.

Hint: for any $x \in X$ using an appropriate open affine neighborhood $x \in \operatorname{Spec} A \subset$ such that $\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B \subset Y$, construct an open affine neighborhood of $\Delta(x)$ in $X \times_Y X$.

Exercise 2.9.19. Let $U \subset \operatorname{Spec} A$ be an open subscheme of an affine scheme. Prove that U is separated.

Exercise 2.9.20. (Solution at p. 109) Consider the following three conditions on a morphism of schemes:
(1) f is proper.

(2) f is finite,

(3) f is separated,
For each $1 \leqslant i \neq j \leqslant 3$, state (without proof) whether the implication

$$(i) \Rightarrow (j)$$

holds. If it does not hold, give a counter-example. (I.e., in total you should discuss 6 implications.)

Exercise 2.9.21. We say that a property P of morphisms of schemes is *stable under left cancellation* if

$$P(g \circ f) \text{ and } P(g) \Rightarrow P(f)$$

for any composable morphisms (i.e., $X \xrightarrow{f} Y \xrightarrow{g} Z$). Prove that the properties "f is proper" and "f is separated" are stable under left cancellation.

More sharply, show the following assertions:

- "$g \circ f$ proper" and "g separated" $\Rightarrow$ "f proper".
- "$g \circ f$ separated" $\Rightarrow$ "f separated".

Exercise 2.9.22. Let k be an algebraically closed field. Let X be a scheme that is connected and proper over $\operatorname{Spec} k$. Prove that any morphism $f : X \to \mathbf{A}^1_k$ is constant.

Hint: using Exercise 2.9.21, what can you say about the image of the composite $X \xrightarrow{f} \mathbf{A}^1_k \subset \mathbf{P}^1_k$?

Exercise 2.9.23. (Solution at p. 109) For the purposes of this exercise we call a morphism $f : Y := \operatorname{Spec} B \to X := \operatorname{Spec} A$ *nice* if the diagonal map

$$\Delta : Y \to Y \times_X Y$$

is an *open* immersion.

(1) Which of the following maps are nice (and why, respectively why not)? Which of these maps are flat (and why, respectively why not)?
   - $\operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$, for some $f \in A$,
   - $\operatorname{Spec} \mathbf{F}_p \to \operatorname{Spec} \mathbf{Z}$,
   - the structural map $\mathbf{A}^1 \to \operatorname{Spec} \mathbf{Z}$,
   - $\operatorname{Spec} \mathbf{C} \to \operatorname{Spec} \mathbf{R}$.

(2) Prove that the composite of two nice maps is nice.

Remark 2.9.24. A morphism f as above is called *étale* if 1) f is locally of finite type, 2) B is flat over A and 3) nice in the sense above. A morphism that satisfies 1) and 3) is more commonly referred to as an *unramified* morphism.

2.10 Quasi-coherent sheaves

For a ring A and $X = \operatorname{Spec} A$, and an A-module M, we have constructed in Lemma 1.5.3 a sheaf $\widetilde{M} \in \operatorname{Shv}(X)$. In this section, we introduce the concept of quasi-coherent sheaves on arbitrary schemes, which can be thought of as being glued together (in the sense of Exercise 1.5.10) from sheaves of the form $\widetilde{M}$. We will prove that quasi-coherent sheaves on affine schemes $X = \operatorname{Spec} A$

are precisely these sheaves $\widetilde{M}$ in the sense that there is an equivalence of categories between Mod_A (the category of A-modules) and the category of quasi-coherent sheaves $\mathrm{QCoh}(\operatorname{Spec} A)$. We start by defining a larger "container" in which quasi-coherent sheaves live. (It is fair to say that the consideration of this larger category is a bit of an artifact though; Exercise 2.10.24 sketches a way to completely avoid it.)

Definition 2.10.1. Let X be a scheme (or even a ringed space). A *sheaf of $\mathcal{O}_X$-modules* is a sheaf F such that

- $F(U)$ is a module over the ring $\mathcal{O}_X(U)$, and
- the restriction maps $F(U) \to F(V)$ is a map of $\mathcal{O}_X(U)$-modules (where $F(V)$ is regarded as an $\mathcal{O}_X(U)$-module via the natural ring map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$).

A morphism of $\mathcal{O}_X$-modules is a sheaf morphism such that all maps $F(U) \to G(U)$ are $\mathcal{O}_X(U)$-linear maps. This defines a category denoted $\mathrm{Mod}_{\mathcal{O}_X}\mathrm{Shv}(X)$ or just $\mathrm{Mod}_{\mathcal{O}_X}$.

(The notation is motivated by the fact that equivalently, one can say that $\mathcal{O}_X$ is a ring object in the category $\mathrm{Shv}(X)$, and an $\mathcal{O}_X$-module is a module object over $\mathcal{O}_X$ etc.)

Example 2.10.2. The sheaf $\widetilde{M} \in \mathrm{Shv}(X)$, $X = \operatorname{Spec} A$ mentioned above is a sheaf of $\mathcal{O}_X$-modules. Indeed, by using the equivalent description of sheaves from Lemma 1.5.2, it suffices to check the $\mathcal{O}_X(U)$-module structure for $U = D(f)$ only. In this case $\widetilde{M}(D(f)) = M[f^{-1}]$ is indeed an $\mathcal{O}_X(D(f)) = A[f^{-1}]$-module etc. Also, given a map $M \to N$ of A-modules, the associated map $\widetilde{M} \to \widetilde{N}$ is clearly a map of $\mathcal{O}_X$-modules. This defines a functor

$$\widetilde{-} : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}_X}.$$

Lemma 2.10.3. For an $\mathcal{O}_X$-module F, we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathcal{O}_X, F) \stackrel{\cong}{\to} F(X).$$

Proof. The right hand side consists of compatible collections of maps, for each open $U \subset X$, $\mathcal{O}_X(U) \to F(U)$, each of which is a map of $\mathcal{O}_X(U)$-modules. So this is nothing but an element $f_U \in F(U)$ (namely the image of $1 \in \mathcal{O}_X(U)$). The compatibility amounts to $\mathrm{res}^U_X f_X = f_U$, so such a map is uniquely specified by $f_X \in F(X)$. $\square$

In the proof of Proposition 2.10.6 we will use the following generality:

Lemma 2.10.4. For a ring homomorphism $f : A \to B$, the forgetful functor

$$u : \mathrm{Mod}_B \to \mathrm{Mod}_A$$

(i.e., a B-module M is just regarded as an A-module, by means of $a \cdot m := f(a)m$) admits a left adjoint, given by

$$B \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_B,$$

and a right adjoint, given by

$$\mathrm{Hom}_A(B, -) : \mathrm{Mod}_A \to \mathrm{Mod}_B.$$

Proof. The first claim means that for any B-module N and any A-module N there is a (functorial) bijection

$$\mathrm{Hom}_{\mathrm{Mod}_B}(B \otimes_A M, N) = \mathrm{Hom}_{\mathrm{Mod}_A}(M, u(N)). \tag{2.10.5}$$

This is precisely the universal property of the tensor product. The one for Hom (which is not used below) also follows from such considerations, see, e.g. [Eis95, §A5.2] for more background. $\square$

Proposition 2.10.6. Let A be a ring and $X = \operatorname{Spec} A$.

(1) There is an adjunction
$$\widetilde{-} : \mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_{\mathcal{O}_X} : \Gamma,$$
where as before Γ is the global sections functor, i.e., $\Gamma(F) = F(X)$ (which is an A-module).

(2) The functor $\widetilde{-}$ is fully faithful.

(3) For $F \in \mathrm{Mod}_{\mathcal{O}_X}$, the following are equivalent:

(a) F lies in the essential image of $\widetilde{-}$.

(b) X admits an open covering $X = \bigcup D(f_i)$ by basic opens such that there is an isomorphism (of $\mathcal{O}_X$-modules)
$$F|_{D(f_i)} = \widetilde{F(D(f_i))}.$$

(c) X admits an open covering by affine subschemes $\operatorname{Spec} B \subset X$ such that there exists an isomorphism (of $\mathcal{O}_X$-modules)
$$F|_{\operatorname{Spec} B} \cong \widetilde{F(\operatorname{Spec} B)}.$$

Proof. (1): We have to show that applying Γ yields a bijection (for any $M \in \mathrm{Mod}_A$ and $F \in \mathrm{Mod}_{\mathcal{O}_X}$)
$$\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_X}}(\widetilde{M}, F) \to \mathrm{Hom}_{\mathrm{Mod}_A}(M, F(X)). \tag{2.10.7}$$

Applying this to $F = \widetilde{N}$ we immediately get the full faithfulness of $\widetilde{-}$, i.e., (2).

This proof is somewhat similar to the one of Theorem 1.6.14. We check the map (2.10.7) is injective. Given two morphisms of $\mathcal{O}_X$-modules $\varphi, \varphi' : \widetilde{M} \to F$ such that $\varphi(X) = \varphi'(X) : M \to F(X)$, we consider the basic open subset $U = D(f)$ and the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\mathrm{res}"] \arrow[d, "g(X)"] & M[f^{-1}] \arrow[d, "\varphi(U)"] \\ F(X) \arrow[r, "\mathrm{res}"] & F(U). \end{tikzcd}$$

Recall that $M[f^{-1}] = M \otimes_A A[f^{-1}]$, and an $A[f^{-1}]$-linear map from here to $F(U)$ (which is an $A[f^{-1}]$-module!) is uniquely determined by its composite with $M \to M[f^{-1}]$. Therefore $\varphi(U) = \varphi'(U)$. By Lemma 1.5.2, $\varphi = \varphi'$.

We check that the map (2.10.7) is surjective. Fix an A-linear map $\varphi : M \to F(X)$. Our goal is to extend this to a map of $\mathcal{O}_X$-modules $\widetilde{M} \to F$, so we pick $U = D(f)$, for $f \in A$, and try to define the dotted map

$$\begin{tikzcd} M \arrow[r, "\mathrm{res}"] \arrow[d, "\varphi"] & M[f^{-1}] = M \otimes_A A[f^{-1}] \arrow[d, dotted] \\ F(X) \arrow[r, "\mathrm{res}"] & F(U). \end{tikzcd}$$

Since F is an $\mathcal{O}_X$-module, $F(U) \in \mathrm{Mod}_{A[f^{-1}]}$, so the bijection (2.10.5) ensures the existence and unicity of the dotted map, which we denote $\varphi(U)$. Given a basic open subset $V = D(g) \subset D(f)$, the map $\varphi(V)$ is compatible with $\varphi(U)$ under further restriction, so again invoking Lemma 1.5.2 we have succeeded in constructing the map $\widetilde{M} \to F$.

(3): generally, if for some affines $V = \operatorname{Spec} B[b^{-1}] \subset U = \operatorname{Spec} B \subset X$, $F|_U = \widetilde{F(U)}$, then also $F|_V = \widetilde{F(V)}$ (by construction of the sheaves $\widetilde{-}$, cf. Lemma 1.5.3). Thus, (3)a $\Rightarrow$ (3)b $\Rightarrow$ (3)c. The proof of the implication (3)c $\Rightarrow$ (3)a hinges on the following assertion:

Lemma 2.10.8. Let $U := D(f) \subset X = \operatorname{Spec} A$. Suppose $F \in \operatorname{Mod}_{\mathcal{O}_X}$ satisfies the condition in (3)c. Putting $M := F(X) \in \operatorname{Mod}_A$, there is an isomorphism

$$M[f^{-1}] \overset{\cong}{\to} F(U).$$

Proof. More precisely, we show that
(1) We have an exact sequence

$$0 \to \{m \in M, f^n m = 0\} \to M = F(X) \overset{\text{res}}{\to} F(U).$$

(2) For any $t \in F(U)$ there is some $n \gg 0$ such that $f^n t \in \operatorname{im} \operatorname{res}$.

By quasi-compactness (Lemma 1.1.10), we can pick a finite covering of X by affine opens $V = \operatorname{Spec} B$ with the property that $F|_V = \widetilde{M}$ for $M \in \operatorname{Mod}_B$. By refinining these V, we may assume they are of the form $V_i = D(g_i)$ for $g_i \in A$, say $F(D(g_i)) = M_i$. We have a commutative diagram whose rows are exact by the definition of the left hand term, and whose columns are exact by the sheaf property of F:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \ker \operatorname{res} \arrow[r] \arrow[d] & M = F(X) \arrow[r, "\text{res}"] \arrow[d] & F(U) \arrow[d] \\
0 \arrow[r] & \prod_i \{m_i \in M_i, \exists n_i \gg 0 : f^{n_i} m_i = 0\} \arrow[r] & \prod_i F(D(g_i)) = M_i \arrow[r, "\text{res}"] \arrow[d] & \prod_i F(D(fg_i)) = M_i[f^{-1}] \arrow[d] \\
 & & \prod_{i,j} F(D(g_i g_j)) \arrow[r] & F(D(fg_i g_j))
\end{tikzcd}$$

Since there are finitely many i only, the bottom left kernel agrees with the collection of (m_i) such that $f^n m_i = 0$ for $n \gg 0$. Now, if $m \in \ker \operatorname{res}$, using the exactness of the middle row, we obtain $f^n m = 0$ for $n \gg 0$.

The restriction of $t \in F(U)$ to $U \cap V_i$ is an element of $F(D(fg_i)) = M_i[f^{-1}]$. Again using the finiteness of the covering, there is some $n \gg 0$ such that $f^n m_i \in M_i$. The element $f^n(m_i - m_j) \in F(D(g_i g_j))$ may not be zero, but its restriction to $F(D(fg_i g_j))$ is zero, so there is again a uniform $m \gg n$ such that $f^m(m_i - m_j) = 0$ by the first part. Thus $f^m t$ lies in the image of the restriction map. ☐

We now prove the remaining implication (3)c $\Rightarrow$ (3)a, using Lemma 2.10.8 and the following generality from category theory: a functor $F : C \to D$ is an equivalence of categories if (and only if) a) F admits a right adjoint G, b) F is fully faithful, c) G is conservative. (The full faithfulness of F equivalent to the unit map $u : \mathrm{id}_C \to GF$ being an isomorphism; the counit $c : FG \to \mathrm{id}_D$ is an isomorphism as well since $G(c) : GFG \to G$ agrees with uG, then use the conservativity of G).

Now, we have checked that $\widetilde{-}$ is fully faithful in (2); the conservativity of its right adjoint Γ (on the full subcategory of $\mathcal{O}_X$-modules satisfying the condition in (3)c) holds by Lemma 2.10.8: some map $\varphi : F \to G$ in $\operatorname{QCoh}(X)$ is an isomorphism iff $\varphi(D(f)) : F(D(f)) \to G(D(f))$ is an isomorphism (Lemma 1.5.2), but this map agrees with $\varphi(X)[f^{-1}]$. ☐

Definition 2.10.9. Let X be a scheme. A *quasi-coherent sheaf* on X is an $\mathcal{O}_X$-module F with the property that there is some covering $X = \bigcup U_i = \bigcup \operatorname{Spec} A_i$ by open affine subschemes such that

$$F|_{U_i} \cong \widetilde{F(U_i)}.$$

A morphism of quasi-coherent sheaves is, by definition, an $\mathcal{O}_X$-linear sheaf homomorphism. In other words, the objects F satisfying the condition above form a full subcategory, denoted $\operatorname{QCoh}(X)$, of $\operatorname{Mod}_{\mathcal{O}_X} \operatorname{Shv}(X)$.

Using Corollary 2.10.10, one can equivalently replace the existence of *some* covering with the above property by the condition that $F|_{\operatorname{Spec} A} = \widetilde{F(\operatorname{Spec} A)}$ for *any* open affine $\operatorname{Spec} A \subset X$. In particular:

Corollary 2.10.10. For any ring A and $X = \operatorname{Spec} A$, there is an equivalence of categories

$$\tilde{-} : \mathrm{Mod}_A \rightleftarrows \mathrm{QCoh}(X) : \Gamma.$$

More generally, we have the following description of quasi-coherent sheaves, which avoids the reference to the ambient category of $\mathcal{O}_X$-modules.

Corollary 2.10.11. Let X be a separated scheme, $X = \bigcup \operatorname{Spec} A_i$ a covering by open affines, and let $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j = \operatorname{Spec} A_{ij}$ (Lemma 2.9.3). Then there is an equivalence of categories

$$\mathrm{QCoh}(X) \overset{\cong}{\to} \{(M_i \in \mathrm{Mod}_{A_i}, \phi_{ij} : M_i \otimes_{A_i} A_{ij} \overset{\cong}{\to} M_j \otimes_{A_j} A_{ij}) \mid \phi_{jk} \circ \phi_{ij} = \phi_{ij} : M_i \otimes_{A_i} A_{ij} \otimes_{A_j} A_{jk} \to M_j \otimes_{A_i} A_{ij} \otimes_{A_j}$$

Proof. Given a collection of (M_i, ϕ_{ij}) at the right hand side (satisfying the cocycle condition), we consider the associated quasi-coherent sheaves $\widetilde{M_i} \in \mathrm{QCoh}(\operatorname{Spec} A_i)$ and $\widetilde{\phi_{ij}}$ (in $\mathrm{QCoh}(\operatorname{Spec} A_{ij})$). By Exercise 1.5.10, there is a unique sheaf F on X whose restrictions are isomorphic to $\widetilde{M_i}$. This is an $\mathcal{O}_X$-module since the $\widetilde{M_i}$ are. It is quasi-coherent by Definition 2.10.9.

Conversely, for $F \in \mathrm{QCoh}(X)$, $F_i := F|_{\operatorname{Spec} A_i}$ comes equipped with isomorphisms $\phi_{ij} : F_i|_{\operatorname{Spec} A_i \cap \operatorname{Spec} A_j} \cong F_j|_{\operatorname{Spec} A_i \cap \operatorname{Spec} A_j}$ satisfying the cocycle condition. Now, F_i is quasi-coherent on $\operatorname{Spec} A_i$, so $F_i = \widetilde{M_i}$ and the isomorphisms ϕ_{ij} can be expressed by what they do on the sections of $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$. □

Remark 2.10.12. The separatedness above was imposed only to avoid discussing further affine coverings of $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$.

Definition and Lemma 2.10.13. Let A be a graded ring, and consider $X = \operatorname{Proj} A$. Let M be a graded A-module, by which we mean that $M = \bigoplus_{d \geqslant 0} M_d$ and the A-module structure is such that $A \times M \to M$ restricts to $A_m \times M_n \to M_{m+n}$. Morphisms of graded modules are required to preserve the graded components. This defines a category grMod_A. Then there is a functor

$$\tilde{-} : \mathrm{grMod}_A \to \mathrm{QCoh}(\operatorname{Proj} A)$$

characterized by the property (for $f \in A_+$ being homogeneous)

$$(\widetilde{M})|_{D_+(f)} = \widetilde{M[f^{-1}]_0},$$

where as before 0 denotes the elements of degree 0.

Proof. The existence of a sheaf $\widetilde{M}$ with these properties follows from the construction of $\operatorname{Proj} A = \bigcup D_+(f)$. It is quasi-coherent by its definition. □

For example, $\widetilde{A} = \mathcal{O}_{\operatorname{Proj} A}$.

Definition 2.10.14. For a graded A-module M, the *Serre twist* $M(e)$, where $e \in \mathbf{Z}$, is the graded A-module defined by

$$(M(e))_d = M_{e+d}.$$

We denote by $\mathcal{O}_{\operatorname{Proj} A}(e)$ or just $\mathcal{O}(e) := \widetilde{A(e)}$. This sheaf is referred to as the *Serre twist* of the structural sheaf $\mathcal{O}$.

We have the following extension of Lemma 2.3.11.

Lemma 2.10.15. Let $A = B[t_0, \dots, t_n]$, so that $X := \operatorname{Proj} A = \mathbf{P}^n_B$. We have an isomorphism

$$\Gamma(\mathbf{P}^n_B, \mathcal{O}_X(e)) = B[t_0, \dots, t_n]_e,$$

where the subscript e denotes the B-submodule consisting of homogeneous polynomials of degree e (and $\deg t_i = 1$). In particular:

- the global sections vanish for $e < 0$,
- the global sections are a finitely generated B-module for any e (in fact finite free of rank $\binom{n+e}{n}$)

Lemma 2.10.16. For any scheme X, $\operatorname{QCoh}(X)$ is an abelian category (so, in particular, kernels and cokernels exist). Also, there is a tensor product functor

$$\otimes := \otimes_{\mathcal{O}_X} : \operatorname{QCoh}(X) \times \operatorname{QCoh}(X) \to \operatorname{QCoh}(X),$$

where $(F \otimes G)|_{\operatorname{Spec} A} = \widetilde{F(A) \otimes_A G(A)}$, for two quasi-coherent sheaves F, G and $\operatorname{Spec} A \subset X$ open affine.

This functor equips the category with the structure of a symmetric monoidal category (i.e., there is a unit object, namely $\mathcal{O}_X$ for $\otimes$, the tensor product is associative and commutative; see, e.g. [Mac98, §VII] for more background).

Proof. Using general theory of sheaves, one proves that $\operatorname{Shv}(X, \operatorname{Mod}_{\mathbf{Z}})$ (the category of sheaves of abelian groups) is abelian and has a tensor product. This formally implies the same properties for $\operatorname{Mod}_{\mathcal{O}_X}$, see, e.g. [KS05, Theorem 18.1.6]. One then checks that for a map $F \to G$ in $\operatorname{QCoh}(X) \subset \operatorname{Mod}_{\mathcal{O}_X}$, $\ker f$ and $\operatorname{coker} f$, taken in the larger category $\operatorname{Mod}_{\mathcal{O}_X}$, actually lie in $\operatorname{QCoh}(X)$, and are therefore kernel and cokernel in here.

Concerning the tensor product, one can argue similarly, by first establishing a tensor product for $\mathcal{O}_X$-modules. See, e.g. [KS05, §18.2]. The key point in showing that $\operatorname{QCoh}(X) \subset \operatorname{Mod}_{\mathcal{O}_X}$ is stable under tensor product is the following claim: for $X = \operatorname{Spec} A$ affine, and $M, N \in \operatorname{Mod}_A$, we have a natural isomorphism

$$(\widetilde{M \otimes_A N}) \xrightarrow{\cong} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}.$$

Indeed, a map (of $\mathcal{O}_X$-modules) arises from the adjunction established in Corollary 2.10.10: $M \otimes_A N \to \Gamma(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})$ arises by observing that for $(m, n) \in M \times N = \Gamma(\widetilde{M} \times \widetilde{N})$, we have a global section of the presheaf tensor product $\widetilde{M} \otimes^{\mathrm{PSh}}_{\mathcal{O}_X} \widetilde{N}$, and therefore of its sheafification as well. To show the map is an isomorphism it suffices to see it induces an isomorphism on stalks, and by general sheaf theory we have

$$(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})_{\mathfrak{p}} = (\widetilde{M})_{\mathfrak{p}} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} (\widetilde{N})_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}},$$

which agrees with $(M \otimes_A N)_{\mathfrak{p}}$.

Remark 2.10.17. A different (but equivalent) perspective is to use the description of $\operatorname{QCoh}(X)$ in Corollary 2.10.11. For example, the tensor product of quasi-coherent sheaves $F \otimes_{\mathcal{O}_X} F'$ just corresponds to the tensor products $M_i \otimes_{A_i} M'_i$ etc.

Pullback and pushforward

Recall that for a continuous map $f : X \to Y$ (between two topological spaces), we have the *direct image functor* (also called the *pushforward functor*)

$$f_* : \operatorname{Shv}(X) \to \operatorname{Shv}(Y).$$

It is given by $(f_*F)(V) = F(f^{-1}(V))$ (for $V \subset Y$ open, $F \in \mathrm{Shv}(X)$). This functor admits a left adjoint, called the *inverse image functor* or *pullback functor*

$$f^{-1} : \mathrm{Shv}(Y) \to \mathrm{Shv}(X).$$

Among various characterizations, it can be described to be the unique functor that preserves colimits, and that sends the representable sheaves (for $V \subset Y$ open) $h_V : U \mapsto \mathrm{Hom}_Y(U, V)$ to $h_{f^{-1}(V)}$. See, e.g., [KS05, §17.5]. However, all properties of f^{-1} can be deduced solely from this being the left adjoint of f_*, cf. Exercise 2.10.22 for an illustration of the principle.

Suppose now that $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_X)$ is a map of ringed spaces (for example a morphism of schemes). Then the above functors give adjoint functors

$$f^* : \mathrm{Mod}_{\mathcal{O}_Y}\mathrm{Shv}(Y) \rightleftarrows \mathrm{Mod}_{\mathcal{O}_X}\mathrm{Shv}(X) : f_*,$$

where f_* is the functor above, which uses the $\mathcal{O}_Y$-action on f_*F

$$\mathcal{O}_Y \times f_*F \xrightarrow{f^\sharp} f_*\mathcal{O}_X \times f_*F = f_*(\mathcal{O}_X \times F) \xrightarrow{f_*(\mathrm{act})} f_*F.$$

It is a formal (i.e., category-theoretic) consequence of the setup that the left adjoint is given by

$$f^*F = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}F.$$

For example,

$$f^*\mathcal{O}_Y = \mathcal{O}_X. \tag{2.10.18}$$

Here is how to connect quasi-coherent sheaves on different schemes. We say that a map $f : X \to Y$ of schemes is *quasi-separated* if $\Delta_F : X \to X \times_Y X$ is a quasi-compact morphism. (This condition can be checked locally on Y; if $Y = \mathrm{Spec}\, A$ is affine, then X is quasi-separated iff for any open affines $U, V \subset X$, $U \cap V$ admits a *finite* covering by affine open subschemes.)

Lemma 2.10.19. Let $f : X \to Y$ be a map of schemes.
(1) The pullback functor $f^* : \mathrm{Mod}_{\mathcal{O}_Y} \to \mathrm{Mod}_{\mathcal{O}_X}$ preserves quasi-coherent sheaves, i.e., it restricts to a functor

$$f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X).$$

(2) Suppose f is quasi-compact (Definition and Lemma 2.7.1) and quasi-separated. (For example, this is true whenever X is Noetherian and f and Y arbitrary.) Then the pushforward functor $f_* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$ preserves quasi-coherent sheaves, i.e., in this case we have an adjunction

$$f^* : \mathrm{QCoh}(Y) \rightleftarrows \mathrm{QCoh}(X) : f_*.$$

Example 2.10.20. Suppose $f : X = \mathrm{Spec}\, B \to Y = \mathrm{Spec}\, A$, for a ring homomorphism $A \to B$. We claim that the adjunction

$$f^* : \mathrm{QCoh}(Y) \rightleftarrows \mathrm{QCoh}(X) : f_*$$

is, under the equivalence with the categories of modules, simply given by

$$f^* = B \otimes_A - : \mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_B : f_* = \mathrm{forget}.$$

Indeed, for any A-module M, there is a resolution

$$\bigoplus_{i \in I} A \to \bigoplus_{j \in J} A \to M \to 0.$$

(with generally infinite sets I, J). Using the equivalence Corollary 2.10.10 for Y, we have an exact sequence

$$\bigoplus_{i \in I} \widetilde{A} \to \bigoplus_{j \in J} \widetilde{A} \to \widetilde{M} \to 0.$$

The functor f^* is right-exact. Applying f^* gives (by (2.10.18)) an exact sequence in $\mathrm{QCoh}(X)$

$$\bigoplus_{i\in I}\widetilde{B}\to\bigoplus_{j\in J}\widetilde{B}\to f^*(\widetilde{M})\to 0. \tag{2.10.21}$$

However, applying $B\otimes_A -$ (which is right exact) to the first exact sequence, we get

$$\bigoplus_{i\in I}B\to\bigoplus_{j\in J}B\to B\otimes_A M\to 0.$$

Using the equivalence $\mathrm{Mod}_B=\mathrm{QCoh}(X)$, we see that

$$f^*(\widetilde{M})=\widetilde{M\otimes_A B}.$$

We conclude from this and from Lemma 2.10.4 that f_* is the forgetful functor.

Proof. The condition of being a quasi-coherent sheaf, and the condition of being a qcqs map are local (on Y, Lemma 2.8.2), so we may assume $Y=\operatorname{Spec}A$ is affine.

(1): it is enough to show $f^*(\widetilde{M})\in\mathrm{QCoh}(X)$ for any $M\in\mathrm{Mod}_A$, but this is true by virtue of an exact sequence as in (2.10.21), noting that $f^*(\widetilde{A})=f^*\mathcal{O}_Y=\mathcal{O}_X$.

(2): By assumption X is quasi-compact and quasi-separated, so X has a finite covering $X=\bigcup\operatorname{Spec}B_i$, $i\in I$ by open affines. Since X is quasi-separated, $\operatorname{Spec}B_i\cap\operatorname{Spec}B_j=\bigcup_{k\in I_{ij}}\operatorname{Spec}B_{ijk}$ is again a finite covering. As a preliminary observation, note that for any sheaf F on X and any open $U\subset X$, we have an exact sequence

$$0\to F(U)\to\prod_{i\in I}F(\operatorname{Spec}B_i\cap U)\to\prod_{i,j\in I}\prod_{k\in I_{ij}}F(\operatorname{Spec}B_{ijk}\cap U).$$

Indeed, this holds since the maps from $F(\operatorname{Spec}B_i\cap\operatorname{Spec}B_j)\to\prod_{k\in I_{ij}}F(\operatorname{Spec}B_{ijk})$ are injective (by the sheaf condition).

Let $F\in\mathrm{QCoh}(X)$ and let $M:=\Gamma(f_*F)=F(X)\in\mathrm{Mod}_B$ be the global sections of f_*F. We need to show that for any $a\in A$ and $V:=D(a)\subset Y$, there is an isomorphism

$$M[a^{-1}]=f_*F(V)=F(f^{-1}(V)).$$

We apply the above exact sequence to $U=X$:

$$0\to F(X)=M\to\prod_i F(\operatorname{Spec}B_i)\to\prod_{i,j,k}F(\operatorname{Spec}B_{ijk}).$$

These products are *finite*, so localizing (which is an exact functor) gives an exact sequence

$$0\to M[a^{-1}]\to\prod_i(F(\operatorname{Spec}B_i)[a^{-1}])\to\prod_{i,j,k}(F(\operatorname{Spec}B_{ijk})[a^{-1}]).$$

We also apply the above exact sequence to $U=f^{-1}(V)$:

$$0\to F(f^{-1}(V))\to\prod_i F(\operatorname{Spec}B_i\cap f^{-1}(V))\to\prod_{i,j,k}F(\operatorname{Spec}B_{ijk}\cap f^{-1}(V)).$$

We see that the terms in the two right hand products agree ($\operatorname{Spec}B_i\cap f^{-1}(V)$ is the preimage of V under the map $\operatorname{Spec}B_i\to Y$, i.e., it is $\operatorname{Spec}B_i[a^{-1}]$; then use that F is quasi-coherent.) $\square$

Exercises

Exercise 2.10.22. Let $f : X \to Y$ be a continuous map. Let $f^? : \mathrm{Shv}(Y, \mathrm{Mod}_{\mathbf{Z}}) \to \mathrm{Shv}(X, \mathrm{Mod}_{\mathbf{Z}})$ be a left adjoint of f_*. (It was asserted above that f^* is the left adjoint of f_*; the point of the notation $f^?$ and of this exercise is not to use anything claimed above.)

- Let $\mathbf{Z}$ denote the sheaf on a singleton $\{\star\}$ whose global sections are $\mathbf{Z}$. Let us write $\mathbf{Z}_X := p_X^? \mathbf{Z}$, where $p_X : X \to \{\star\}$ is the unique map. Prove that $f^? \mathbf{Z}$ is the so-called *constant sheaf*, which is given by
$$f^? \mathbf{Z}(U) = \mathbf{Z}^{\pi_0(U)}.$$
(A direct sum of copies of $\mathbf{Z}$, one for each connected component of U.)
Hint: prove that this is a sheaf on X (while the constant presheaf $U \mapsto \mathbf{Z}$ is generally not a sheaf). Then check that
$$\mathrm{Hom}_{\mathrm{Shv}(X)}(f^? \mathbf{Z}, F) = \mathrm{Hom}(\mathbf{Z}, (p_X)_* F).$$
- Prove $f^? \mathbf{Z}_Y = \mathbf{Z}_X$.
- For a point $x \in X$, let $i : \{\star\} \to X$ be the map that sends $\star$ to x. Show that for $F \in \mathrm{Shv}(X)$ (or also $\mathrm{Shv}(X, \mathrm{Mod}_{\mathbf{Z}})$ etc.) $i^? F = F_x$ (the stalk of F). Deduce that
$$(f^? F)_x = F_{f(x)}.$$

Exercise 2.10.23. Let $j : \mathbf{G}_\mathrm{m} \to \mathbf{A}^1$ be the standard open immersion. Consider the *extension by zero* of $\mathcal{O}_{\mathbf{G}_\mathrm{m}}$, defined as
$$(j_! \mathcal{O}_{\mathbf{G}_\mathrm{m}})(U) = \begin{cases} \mathcal{O}_{\mathbf{G}_\mathrm{m}}(U) & U \subset \mathbf{G}_\mathrm{m} \\ 0 & \text{else} \end{cases}$$
Prove that this is an $\mathcal{O}_{\mathbf{A}^1}$-module, but that this is *not* a quasi-coherent sheaf.

This example pins down a decisive difference between quasi-coherent sheaves in algebraic geometry and general sheaves in algebraic topology. A beautiful fix to this issue is offered by the recent advent of condensed mathematics [Sch19, esp. §8].

Exercise 2.10.24. This exercise can be regarded as a natural extension of the description of quasi-coherent sheaves in Corollary 2.10.11. Generally, approaching a scheme X by the sets of its A-points (see around Exercise 2.0.3) is known as the "*functor of points approach*", and this exercise stipulates that a quasi-coherent sheaf on a scheme X is ultimately just depending on the points $X(A)$ of the scheme (however, for all rings A, not just fields).

Prove that for a scheme X, there is an equivalence
$$\mathrm{QCoh}(X) \stackrel{\cong}{\to} \lim_{\mathrm{Spec}\, A \xrightarrow{f} X} \mathrm{Mod}_A,$$
where the right hand category is the category whose objects are
$$(M_f \in \mathrm{Mod}_A, \varphi_c : M_f \otimes_A B \stackrel{\cong}{\to} M_g)$$
where M_f is the datum of an A-module for any map $\mathrm{Spec}\, A \to X$ (we do not insist $\mathrm{Spec}\, A$ to be an open subscheme of X). At the right, for a commutative diagram
$$\begin{tikzcd} \mathrm{Spec}\, B \arrow[rr, "c"] \arrow[dr, "g"'] & & \mathrm{Spec}\, A \arrow[dl, "f"] \\ & X & \end{tikzcd}.$$

Finally, the φ_C are subject to the condition that the composite

$$M_f \otimes_A B \otimes_B C \stackrel{\varphi_c \otimes \mathrm{id}_C}{\to} M_g \otimes_B C \stackrel{\varphi_d}{\to} M_h$$

should be equal to φ_{dc}, where

$$\begin{tikzcd} \operatorname{Spec} C \arrow[rr, "d"] \arrow[dr, "h"'] & & \operatorname{Spec} B . \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Morphisms $(M_f, \varphi_c) \to (M'_f, \varphi'_c)$ are maps $M_f \to M'_f$ in Mod_A that are compatible with φ_c and φ'_c in the obvious sense.

Hint: the key geometric input is the idea (for arbitrary $f : \operatorname{Spec} A \to X$) to choose an open covering $\operatorname{Spec} A$ of open affines whose images under f are contained in an affine subscheme of X.

Exercise 2.10.25. Let $f : X \to Y$ be a map of schemes. Show that $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is given by the functor

$$\lim_{\operatorname{Spec} A \stackrel{y}{\to} Y} \mathrm{Mod}_A \to \lim_{\operatorname{Spec} A \stackrel{x}{\to} X} \mathrm{Mod}_A$$

sending (M_y, φ_c) to the collection whose component for $x : \operatorname{Spec} A \to X$ is simply $M_{f\circ x}$.

Exercise 2.10.26. For a map $f : X \to Y$ of schemes, prove that

- f^* preserves colimits (i.e., the natural map $\operatorname{colim}_i f^*(F_i) \to f^*(\operatorname{colim} F_i)$ is an isomorphism),
- f^* preserves the tensor product, i.e., there are natural isomorphisms

$$f^*(M \otimes_{\mathcal{O}_Y} M') \stackrel{\cong}{\to} f^*M \otimes_{\mathcal{O}_X} f^*M'.$$

These turn out to be the critical two properties of f^*. Indeed, [BC14] proves that for two quasi-compact quasi-separated schemes X, Y there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Sch}}(X, Y) \stackrel{\cong}{\to} \mathrm{Fun}^{\mathrm{colim},\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X)), f \mapsto f^*$$

where at the right one considers functors that preserve colimits and are compatible with the tensor product!

Exercise 2.10.27. For a quasi-compact and quasi-separated map $f : X \to Y$ and $F \in \mathrm{QCoh}(X)$, $G \in \mathrm{QCoh}(Y)$, establish an isomorphism

$$f_*F \otimes G \stackrel{\cong}{\to} f_*(F \otimes f^*G).$$

This isomorphism is referred to as the *projection formula*.

Exercise 2.10.28. Prove the following analogue of Lemma 2.10.8 on projective space $X := \mathbf{P}^n_B = \operatorname{Proj} B[t_0, \dots, t_n]$. Let $F \in \mathrm{QCoh}(X)$. Let us write $F(\star) := \bigoplus_{e\in\mathbf{Z}} F(e) (\in \mathrm{QCoh}(X))$ and $M := \Gamma(X, F(\star)) = \bigoplus_{e\in\mathbf{Z}} \Gamma(X, F(e))$. Let $U = D_+(t_i)$. Establish an isomorphism of graded $B[t_0, \dots]$-modules

$$M[t_i^{-1}] = (F(\star))(U).$$

In particular, for any $s \in F(U) \subset F(\star(U))$, there is some $d \gg 0$ such that $t_i^d s (\in F(d)(U))$ extends to a global section of $F(d)$.

Exercise 2.10.29. Fix a scheme X and a morphism $F \to G$ of quasi-coherent sheaves. For an affine open $U \subset X$, let $P(U)$ be the property "$F(U) \to G(U)$ is surjective" (resp. injective, resp. bijective). Prove that this property is affine-local (as defined in Proposition 2.1.7).

2.11 Line bundles and vector bundles

Definition 2.11.1. A quasi-coherent sheaf F on a scheme X is called a *vector bundle* (resp. *line bundle*) if X admits a covering by open affines $\operatorname{Spec} A$ such that there is an isomorphism (of quasi-coherent sheaves)

$$F|_{\operatorname{Spec} A} \cong \mathcal{O}^n_{\operatorname{Spec} A}$$

(resp. $F|_{\operatorname{Spec} A} \cong \mathcal{O}_{\operatorname{Spec} A}$).

A *trivial vector bundle* (resp. *trivial line bundle*) is one isomorphic to ${\mathcal{O}_X}^n$ (resp. $\mathcal{O}_X$).

We denote by

$$\operatorname{Vb}(X) := \{\text{vector bundles}\}/\cong$$

the set of vector bundles up to isomorphism. This is an (abelian) monoid with respect to the tensor product. The subset

$$\operatorname{Pic}(X) \subset \operatorname{Vb}(X)$$

consisting of the line bundles, is an abelian group: for a line bundle $\mathcal{L}$, the *dual line bundle*

$$\mathcal{L}^\vee := \underline{\operatorname{Hom}}(\mathcal{L}, \mathcal{O}_X)$$

(cf. Exercise 2.11.12) is again a line bundle, and $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \to \mathcal{O}_X$ is an isomorphism.

Lemma 2.11.2. For $X = \operatorname{Spec} A$ and a quasi-coherent sheaf $F = \widetilde{M}$ (for $M \in \operatorname{Mod}_A$) the following are equivalent:

(1) F is a vector bundle,

(2) M is a finite projective A-module,

(3) M is a finitely presented flat A-module,

(4) M is finitely presented and the localizations $M_\mathfrak{p}$ are free $A_\mathfrak{p}$-modules, for all $\mathfrak{p} \in \operatorname{Spec} A$.

Proof. See, e.g. [Stacks, Tag 00NX]. The proof uses the Nakayama lemma (Lemma 1.7.10). □

Lemma 2.11.3. Let A be a PID or a local ring. Then any vector bundle (and thus any line bundle) on $\operatorname{Spec} A$ is trivial:

$$\begin{aligned} \operatorname{Vb}(\operatorname{Spec} A) &= \mathbf{N}, \\ \operatorname{Pic}(\operatorname{Spec} A) &= \{\star\}. \end{aligned}$$

For example, for a field k, any vector bundle on $\mathbf{A}^1_k$ is trivial.

Proof. Our vector bundle V corresponds to a finite projective A-module. We recall from commutative algebra (see, e.g., [Stacks, Tag 0ASV]) that for any PID A, any finitely generated projective A-module is actually *free*. The same holds for local rings by Lemma 2.11.2. □

By contrast, on $\mathbf{P}^n$, we have the Serre twists $\mathcal{O}_{\mathbf{P}^n}(e)$. These are locally (namely on each $D_+(t_i)$) isomorphic to $\mathcal{O}_{\mathbf{P}^n}$, so $\mathcal{O}(e)$ is a line bundle. We have

$$\operatorname{rk} \Gamma(\mathbf{P}^n, \mathcal{O}(e)) = \begin{cases} 0 & e < 0 \\ 1 & e = 0 \\ > 1 & e > 0, n \geqslant 1 \end{cases}$$

(Lemma 2.10.15). Therefore, for $e \neq 0$, $\mathcal{O}(e)$ is *not* trivial (see also Exercise 2.11.13). For the projective line over a field, these are essentially the only outliers, in the following sense.

Theorem 2.11.4. Let k be a field, and V a vector bundle over $X = \mathbf{P}^1_k$. Then there is an isomorphism

$$V = \bigoplus_{i=1}^{m} \mathcal{O}(e_i).$$

Moreover, the number m and the integers e_i are (up to permutation) uniquely determined by V.

In order to prove Theorem 2.11.4 we first establish the following result:

Proposition 2.11.5. There is a bijection between the set $\mathrm{Vb}_n(\mathbf{P}^1_k)$ of rank n vector bundles (up to isomorphism) on $\mathbf{P}^1_k$, up to isomorphism, with the double coset space

$$\mathrm{GL}_n(k[t^{-1}])\backslash \mathrm{GL}_n(k[t^{\pm 1}])/\mathrm{GL}_n(k[t]).$$

Proof. Let V be a vector bundle. We use the covering $\mathbf{P}^1 = D_+(t_0) \cup D_+(t_1) =: U_0 \cup U_1$. Recall that $U_0 = \operatorname{Spec} k[t_0, t_1, t_0^{-1}]_0$ is isomorphic to $\mathbf{A}^1_k = \operatorname{Spec} k[\frac{t_1}{t_0}]$ and, by symmetry $U_1 = \operatorname{Spec} k[t_0, t_1, t_1^{-1}]_0 = \operatorname{Spec} k[\frac{t_0}{t_1}]$. Their intersection, denoted $U_{01} := D_+(t_0 t_1) = \operatorname{Spec} k[t_0, t_1, (t_0 t_1)^{-1}]_0 = \operatorname{Spec}(k[t_0, t_1, t_1^{-1}]_0[(\frac{t_0}{t_1})^{-1}]$ is isomorphic to $\mathbf{G}_\mathrm{m} = \operatorname{Spec} k[v, v^{-1}]$ (where $v = \frac{t_0}{t_1}$). By Lemma 2.11.3, we can choose (for $k = 0, 1$) an isomorphism $\varphi_k : \mathcal{O}^n_{U_k} \xrightarrow{\cong} V|_{U_k}$. Their restrictions to U_{01} give rise to an isomorphism α:

$$\begin{tikzcd} \mathcal{O}^n|_{U_{01}} \arrow[r, "\alpha", "\cong"'] \arrow[d, "\cong"', "\varphi_0|_{U_{01}}"] & \mathcal{O}^n|_{U_{01}} \arrow[d, "\cong"', "\varphi_1|_{U_{01}}"] \\ V|_{U_{01}} \arrow[r, equal] & V|_{U_{01}} \end{tikzcd} \tag{2.11.6}$$

Such an isomorphism (of quasi-coherent sheaves on $\mathbf{G}_\mathrm{m}$) is nothing but an element of $\mathrm{GL}_n(k[t^{\pm 1}])$. The construction of this element depended on the choice of the isomorphisms φ_0, φ_1. Different such isomorphisms are obtained by postcomposing with (the restriction to U_{01} of) an isomorphism $\mathcal{O}^n_{U_1} \cong \mathcal{O}^n_{U_1}$, i.e., by multiplying with an element of $\mathrm{GL}_n(k[t])$, and likewise with U_-. □

Example 2.11.7. We illustrate the above for the line bundle $\mathcal{O}(e)$, with $e \in \mathbf{Z}$. Note that the trivialization of $\mathcal{O}(e)$ on $D_+(t_0)$ is given by multiplication with the unit t_0^e:

$$\mathcal{O}(D_+(t_0)) = \mathbf{Z}[t_0, t_1, t_0^{-1}]_0 \underset{\cong}{\xrightarrow{t_0^e}} \mathbf{Z}[t_0, t_1, t_0^{-1}]_e = \mathcal{O}(e)(D_+(t_0)).$$

Therefore the diagram in 2.11.6 reads

$$\begin{tikzcd} \mathbf{Z}[t_0, t_1, t_0^{-1}, t_1^{-1}]_0 \arrow[r, "\alpha"] \arrow[d, "t_0^e"] & \mathbf{Z}[t_0, t_1, t_0^{-1}, t_1^{-1}]_0 \arrow[d, "t_1^e"] \\ \mathbf{Z}[t_0, t_1, t_0^{-1}, t_1^{-1}]_e \arrow[r, equal] & \mathbf{Z}[t_0, t_1, t_0^{-1}, t_1^{-1}]_e. \end{tikzcd}$$

Thus, α is given by multiplication with $(\frac{t_0}{t_1})^e$.

This reduces the computation of $\mathrm{Vb}(\mathbf{P}^1_k)$ to understanding the double coset space, which is provided by the following elementary statement due to Kronecker and Weber in the 1880's.

Lemma 2.11.8. Let $M \in \mathrm{GL}_n(k[t^{\pm 1}])$ be a matrix whose determinant is t^s, for $s \in \mathbf{Z}$. Then there are matrices $U_+ \in \mathrm{GL}_n(k[t])$, $U_- \in \mathrm{GL}_n(k[t^{-1}])$ such that

$$U_- M U_+ = \operatorname{diag}(t^{r_1}, \dots, t^{r_n}),$$

where $r_1 \geqslant \cdots \geqslant r_n (\in \mathbf{Z})$ are uniquely determined by X.

Proof. For $n = 1$ we have $\mathrm{GL}_1(k[t^{\pm 1}]) = k[t^{\pm 1}]^\times = k^\times \times \mathbf{Z}$, since an invertible element is of the form λt^n, $\lambda \in k^\times$, $n \in \mathbf{Z}$. By contrast, $\mathrm{GL}_1(k[t]) = k[t]^\times = k$. Thus, the double coset space reads

$$\mathrm{GL}_1(k[t^{-1}])\backslash \mathrm{GL}_1(k[t^{\pm 1}])/\mathrm{GL}_1(k[t]) = k^\times\backslash(k^\times \times \mathbf{Z})/k^\times = \mathbf{Z}.$$

Under this bijection, $e \in \mathbf{Z}$ corresponds to $t^e \in \mathrm{GL}_1(k[t^\pm])$. This corresponds to the line bundle $\mathcal{O}_{\mathbf{P}^1}(e)$.

For general n, a more involved, but completely elementary argument using Gaussian elimination is used. See, e.g., [GW20, Lemma 11.50]. □

The above theorem hinges on the triviality of vector bundles on $\mathbf{A}^1_k$. For curves other than $\mathbf{P}^1_k$ (or $\mathbf{A}^1_k$), this will not carry over, but there is the following more local description of vector bundles. If X is a smooth curve over an algebraically closed field k (i.e., irreducible, reduced, of dimension 1, and all the local rings $\mathcal{O}_{X,x}$ are discrete valuation rings), and $K := k(X) = \mathcal{O}_{X,\eta}$ is the function field, i.e., the local ring at the generic point, then one has the *adeles*

$$A_K := \prod_{x \in X \text{ closed}}^{\text{restr.}} Q(\widehat{\mathcal{O}_{X,x}}),$$

where $\widehat{\mathcal{O}_{X,x}}$ denotes the completion of $\mathcal{O}_{X,x}$ at its maximal ideal, and the restricted product means that for all but finitely many closed points x, the entry lies in $\widehat{\mathcal{O}_{X,x}}$. This group of adeles contains the *integral adeles*

$$O_K := \prod_{x \in X \text{ closed}} \widehat{\mathcal{O}_{X,x}}.$$

For any $f \in K$, one can show that $f \in A_K$ (since f has poles only at finitely many points), and therefore $K \subset A_K$. The following theorem, which is a foundational result in the Langlands program, is proved along the lines above, but rather using that a vector bundle on $\operatorname{Spec} K$ and also on $\operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$ is trival; the possible mismatch of these trivializations is accounted for by the appearance of the adeles, with the idea that the intersection of $\operatorname{Spec} K \cap \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$ is $\operatorname{Spec} Q(\widehat{\mathcal{O}_{X,x}})$.

Theorem 2.11.9. (*Weil uniformization theorem*) There is a bijection

$$\mathrm{Vb}_n(X) = \mathrm{GL}_n(K)\backslash \mathrm{GL}_n(A_K)/\mathrm{GL}_n(O_K).$$

Definition 2.11.10. For a scheme X, the *K-group* $K(X)$ (or $K_0(X)$) is the *Grothendieck group* associated to the monoid of vector bundles equipped with the direct sum. That is:

$$K(X) = \bigoplus_{V/X} \mathbf{Z}/[V'] + [V''] - [V],$$

where the direct sum runs over all vector bundles V, and a relation is imposed for any exact sequence

$$0 \to V' \to V \to V'' \to 0.$$

Thus, Theorem 2.11.4 implies

$$K(\mathbf{P}^1_k) = \mathbf{Z} \oplus \mathbf{Z}$$

(one copy for the rank, another one for the twist). For general schemes (even of finite type over k), the computation of K-groups is an open problem!

Example 2.11.11.
- In number theory, on considers rings of algebraic integers $\mathcal{O}_K$, i.e., the integral closure of $\mathbf{Z}$ inside a number field K. These rings are not in general principal ideal domains. The group $\mathrm{Pic}(\mathcal{O}_K)$ is known in this context as the ideal class group. It is known to be a finite group. The *class number formula* uses $\sharp\mathrm{Pic}(\mathcal{O}_F)$ and other data related to K to compute certain values of the ζ-function of K.

  It is also known that any finite projective module over a Dedekind domain A (such as $A = \mathcal{O}_F$) is a direct sum of a free A-module and a line bundle, which leads to $K(\mathcal{O}_F) = \mathbf{Z} \oplus \mathrm{Pic}(\mathcal{O}_F)$.

- One can prove by elementary means, similar to the above that $\operatorname{Pic}(\mathbf{A}^n_k) = 1$ and $\operatorname{Pic}(\mathbf{P}^n_k) = 1$, i.e., all line bundles on these are trivial. See [Stacks, Tag 0BCH] (a line bundle for any UFD, such as $k[t_1, \dots, t_n]$ [Stacks, Tag 0BC1] is trivial) and [Stacks, Tag 0BXJ].
- A much deeper result (due to Quillen and Suslin in the 1970's) shows that any finite projective $k[t_1, \dots, t_n]$-module is free. Thus $\operatorname{Vb}(\mathbf{A}^n_k) = \mathbf{Z}$. But, for $\mathbf{P}^n_k$, there are non-split vector bundles for $n \geqslant 2$. However, at least $K(\mathbf{P}^n_k) = \mathbf{Z}^{n+1}$.

Exercises

Exercise 2.11.12. (1) Let M be a finitely presented A-module and N any A-module. Prove that $\operatorname{Hom}_A(M, N)[f^{-1}] = \operatorname{Hom}_{A[f^{-1}]}(M[f^{-1}], N[f^{-1}])$.

(2) Recall (e.g., from [KS05, §17.7]) that for two $\mathcal{O}_X$-modules F and G, $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(F, G)$ is the sheaf (actually an $\mathcal{O}_X$-module) defined by

$$(\underline{\operatorname{Hom}}_{\mathcal{O}_X}(F, G))(U) := \operatorname{Hom}_{\operatorname{Mod}_{\mathcal{O}_U}}(F|_U, G|_U).$$

Suppose F is a *finitely presented quasi-coherent sheaf* (i.e., X admits a cover by open affines $\operatorname{Spec} A$ such that $F(\operatorname{Spec} A)$ is a finitely presented A-module), and G is any quasi-coherent sheaf. Prove that $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(F, G) \in \operatorname{QCoh}(X)$.

Exercise 2.11.13. Show that a line bundle L on a scheme X is trivial iff there is a global section $s \in L(X)$ such that $s_x \in L_x \cong \mathcal{O}_{X,x}$ is invertible.

Hint: use Lemma 2.10.3.

The following two exercises highlight the categorical aspects of vector and line bundles. See, e.g., [PS13] for an invitation to pervasive topic of dualizability.

Exercise 2.11.14. Prove that a quasi-coherent sheaf F is a vector bundle iff it is a *dualizable object* in the category $\operatorname{QCoh}(X)$.

The latter means that there is another object $G \in \operatorname{QCoh}(X)$ and maps (where here and below all tensor products are over $\mathcal{O}_X$, cf. Lemma 2.10.16)

$$\operatorname{coev} : \mathcal{O}_X \to F \otimes G, \operatorname{ev} : G \otimes F \to \mathcal{O}_X$$

such that the composites

$$F \stackrel{\operatorname{coev}\otimes\operatorname{id}}{\to} F \otimes G \otimes F \stackrel{\operatorname{ev}\otimes\operatorname{id}}{\to} F,$$

$$G \stackrel{\operatorname{id}\otimes\operatorname{coev}}{\to} G \otimes F \otimes G \stackrel{\operatorname{ev}\otimes\operatorname{id}}{\to} G$$

are the identity maps. (Note that this only makes use of the tensor product in $\operatorname{QCoh}(X)$, and the object $\mathcal{O}_X$, which is the *monoidal unit* with respect to this tensor product.)

Hints: first treat the case of $X = \operatorname{Spec} A$ affine, so F corresponds to an A-module M. Use that M being locally free of finite rank is equivalent to being finitely generated projective. If M is dualizable and $\operatorname{coev} : A \to M \otimes N$ sends 1 to the *finite(!)* sum $\sum_{i=1}^n m_i \otimes n_i$, show that the m_i generate M. Using the second identity, prove that the surjection map $A^n \to M$ induced by the m_i admits a splitting.

Exercise 2.11.15. Prove that a quasi-coherent sheaf F is a line bundle iff it is dualizable and the coevaluation (or, equivalently, the evaluation) map is an isomorphism. For this reason, line bundles are also called *invertible sheaves*.

Chapter 3

Cohomology of quasi-coherent sheaves

3.1 Prelude: the Koszul complex

The Koszul complex is a foundational tool in homological algebra and its applications. Textbook accounts of this material include [Eis95, §17], [Wei94, §4.5]; or see [Stacks, Tag 0621]. We will use it below to compute cohomology of affine and projective schemes.

Definition 3.1.1. For an A-module M, the *tensor algebra* is

$$TM := \bigoplus_{n \geqslant 0} M^{\otimes_A n} = A \oplus M \oplus M \otimes_A M \oplus \dots$$

(It is a non-commutative A-algebra whose multiplication is given by juxtaposition of tensors.) The *exterior algebra* $\bigwedge M$ is the non-commutative algebra obtained as the quotient of TM by the two-sided ideal generated by tensors of the form $m \otimes m$, for $m \in M$. The image of a tensor $m_1 \otimes \dots \otimes m_n \in TM$ in $\bigwedge M$ is denoted by $m_1 \wedge \dots \wedge m_n$.

We equip TM and $\bigwedge M$ with the natural grading, i.e., $\bigwedge^n M$ is the image of $M^{\otimes n}$.

For $m, n \in M$, the relation $(m+n) \wedge (m+n) = 0$ can be expanded into

$$m \wedge m + m \wedge n + n \wedge m + n \wedge n = 0,$$

which gives

$$m \wedge n + n \wedge m = 0.$$

Example 3.1.2. If $M = A^{\oplus r}$ is a free A-module of rank r, then TM is the algebra of non-commutative polynomials in r variables. If we denote these variables by $e_1, \dots, e_r$ we have $e_i \wedge e_j = -e_j \wedge e_i$, so that

$$\bigwedge^n M = \bigoplus_{a_1 < \dots < a_n} A e_{a_1} \wedge \dots \wedge e_{a_n}$$

is a free A-module of rank $\binom{r}{n}$. In particular,

$$\bigwedge^0 M = A, \bigwedge^1 M = A^r, \dots, \bigwedge^r M \cong A, \bigwedge^k M = 0 \text{ for } k > r.$$

One refers to $\bigwedge^r M$ as the *determinant* of M.

If, slightly more generally, M is a locally free A module of rank r, then the determinant $\bigwedge^r M$ is locally free of rank 1, and the higher exterior powers vanish.

Definition 3.1.3. Let M be an A-module, and consider an A-module map $\varphi : M \to A$. (In other words, $\varphi \in M^\vee := \operatorname{Hom}_A(M, A)$.) The *Koszul complex* $K(\varphi)$ is the chain complex

$$\dots \to \bigwedge^n M \to \bigwedge^{n-1} M \to \dots \to M \to A,$$

where A is in degree 0 and M in (homological) degree 1 etc. The differential d is the endomorphism of $\bigwedge M$ that is a derivation (of degree -1) and that is given in degree 1 by $d(m) = \varphi(m)$. This means that in degree 2 we have

$$d(m_1 \wedge m_2 \otimes n) = d(m_1) \wedge m_2 \otimes n - m_1 \wedge m_1 d(m_2) \otimes n = (\varphi(m_1)m_2 - \varphi(m_1)m_2) \otimes n$$

and in general

$$d(m_1 \wedge \cdots \wedge m_n \otimes n) = \sum_{i \leqslant n} (-1)^{i+1} \varphi(m_i) e_1 \wedge \cdots \wedge \widehat{m_i} \wedge \cdots \wedge m_n \otimes n.$$

Example 3.1.4. If $M = A$, our map φ is given by an element f, and the complex is

$$K(f) = A \xrightarrow{f} A$$

(in degrees 1 and 0). If $M = A^{\oplus 2}$, and φ corresponds to $f, g \in A$, the complex is given (in degrees 2, 1, and 0) by

$$K(f,g) = Ae_1 \wedge e_2 \xrightarrow{(f,-g)} Ae_1 \oplus Ae_2 \xrightarrow{f,g} A$$

or, more briefly,

$$K(f_1, f_2) = A \xrightarrow{(f_2,-f_1)} A^{\oplus 2} \xrightarrow{f_1,f_2} A$$

We observe that $\mathrm{H}_2(K(f_1, f_2)) = \{a \in A, f_1 a = f_2 a = 0\}$, while $\mathrm{H}_0(K(f_1, f_2)) = A/(f_1, f_2)$.

Definition 3.1.5. In general, if $M = A^{\oplus r}$ is finite free, and φ is given by an ordered n-tuple $\underline{f} = (f_1, \ldots, f_r)$, we also write

$$K(\underline{f}) := K(f_1, \ldots, f_r) := K(\varphi). \tag{3.1.6}$$

We write $K^\vee(\underline{f}) := \mathrm{Hom}_A(K(\underline{f}), A)$ for the cochain complex obtained by taking the termwise dual of $K(\underline{f})$. We refer to it as the *dual Koszul complex*.

Given another A-module N, we also write

$$K(\underline{f}, N) := K(\underline{f}) \otimes_A N$$

$$K^\vee(\underline{f}, N) := \mathrm{Hom}_A(K(\underline{f}), N)$$

(we refer to them as the Koszul complex with coefficients in N and the dual Koszul complex with coefficients in N).

Since $K(\underline{f})$ consists of finitely many, finite free A-modules, we have, for any A-module N,

$$K^\vee(\underline{f}) \otimes_A N = \mathrm{Hom}_A(K(\underline{f}), N).$$

Therefore, the salient homological properties of $K(\underline{f}, N)$ and $K^\vee(\underline{f}, N)$ will follow from the ones of $K(\underline{f})$.

Lemma 3.1.7. In the situation of Definition 3.1.3, fix $m \in M$ and let $a := \varphi(m) \in A$. Then

$$a = dm + md$$

where m denotes the multiplication with m (on $\bigwedge M$; note this raises the degree by 1), while d is the above differential (which lowers the degree by 1).

Proof. Note that dm denotes the endomorphism mapping $n \in \bigwedge M$ to $d(m \wedge n)$, which agrees with $d(m) \wedge n - m \wedge d(n) = \varphi(m)n - m \wedge dn$. $\square$

Corollary 3.1.8. For $f_1, \dots, f_r \in A$, multiplication by each element f_i is null-homotopic on $K(\underline{f})$ and therefore also on $K(\underline{f}, N)$ and $K^\vee(\underline{f}, N)$, for any $N \in \mathrm{Mod}_A$. In particular, all the homology groups $\mathrm{H}_j(K(\underline{f}, N))$ and the cohomology groups $\mathrm{H}^j(K^\vee(\underline{f}, N))$ are annihilated by each f_i and therefore by the ideal $(f_1, \dots, f_r)$. In particular, if $(f_1, \dots, f_r) = A$, then the complexes $K(\underline{f}, N)$, $K^\vee(\underline{f}, N)$ are exact.

Proof. The first part is immediate from the lemma. The statement about homology groups is a generality in homological algebra. (Homotopic maps, in the above situation multiplication by f_i and by 0, induce the same maps on the homology groups; see [Wei94, Lemma 1.4.5] for the very simple proof or [Stacks, Tag 00LO] onwards.) Applying any functor (of chain complexes) to a null-homotopic chain complex gives again a null-homotopic chain complex, which shows that the (dual) Koszul complex is exact also for any N. □

In the following lemma, we use that a cochain complex C^* can be regarded as a chain complex by setting $C_n := C^{-n}$. We apply this to the cochain complex $K^\vee(\underline{f})$, and regard it as a chain complex below. We also need the *shift* of a chain complex $C[p]$, defined by $C[p]_n := C_{p+n}$, with differential $d_{C[p]} := (-1)^p d_C$, cf. [Wei94, p. 1.2.8] for further discussion.

Lemma 3.1.9. For $\underline{f} = (f_1, \dots, f_r)$, there is an isomorphism (of chain complexes of A-modules)

$$K(\underline{f}) \cong K^\vee(\underline{f})[-r]$$

Therefore, for any $N \in \mathrm{Mod}_A$, there are isomorphisms

$$\mathrm{H}_{r-i}(K(\underline{f}, N)) = \mathrm{H}^i(K^\vee(\underline{f}, N)).$$

Proof. If $\underline{f}$ consists of a single element $f \in A$, we have

$$\begin{array}{lccc}
 & \deg 1 & \deg 0 & \deg -1 \\
K(f): & A & \xrightarrow{f} A & \\
K^\vee(f): & & A & \xrightarrow{f} A \\
K^\vee(f)[-1]: & A & \xrightarrow{-f} A &
\end{array}$$

So an isomorphism $K(f) \to K^\vee(f)[-1]$ is given by multiplication with -1 in degree 1 and by the identity in degree 0.

In general, there is an isomorphism of chain complexes $K(\underline{f}) = K(\{f_1\}) \otimes K(\{f_2\}) \otimes \dots \otimes K(\{f_r\})$ and likewise for $K^\vee$, so tensor products of the isomorphisms $K(\{f_i\}) \xrightarrow{\cong} K^\vee(\{f_i\})[-1]$ gives an isomorphism $K(\underline{f}) \cong K^\vee(\underline{f})[-r]$ as requested.

The isomorphism on (co)homology follows (remembering that given a cochain complex C^*, regarded as a chain complex C_* as mentioned above, we have, $\mathrm{H}^n(C^*) = \mathrm{H}_{-n}(C_*)$). □

Definition 3.1.10. Let N be an A-module. A sequence $\underline{f} = (f_1, \dots, f_n)$ of elements of A is called an *almost N-regular sequence* if for each $i \leqslant n$, f_i is a nonzerodivisor in $N/(f_1, \dots, f_{i-1})$.

It is called a *regular sequence* if, in addition, $N/(f_1, \dots, f_n) \neq 0$.

Proposition 3.1.11. If $\underline{f} = (f_1, \dots, f_n)$ is an almost N-regular sequence, then the Koszul complex $K(\underline{f}, N)$ is exact in positive degrees, i.e.,

$$\mathrm{H}_k(K(\underline{f}, N)) = \begin{cases} 0 & k \neq 0 \\ N/(f_i) & k = 0 \end{cases}$$

In other words, we have an exact sequence

$$0 \to \bigwedge^n (A^n) \otimes_A N \to \ldots \to N^n \xrightarrow{f_i} N \to N/(f_i) = N \otimes_A A/(f_i) \to 0.$$

In particular, if $N = A$, this yields a resolution of $A/(f_i)$ by finite free A-modules. In terms of the dual Koszul complex, the above vanishing can be restated as

$$\mathrm{H}_k(K(\underline{f}, N)) = \begin{cases} 0 & k \neq n \\ N/(f_i) & k = n \end{cases} \tag{3.1.12}$$

Proof. If $n = 1$, then $\mathrm{H}_1(K(f, N)) = \ker(N \xrightarrow{f} N) = 0$ since f is nonzerodivisor.

If $n = 2$, we observe that we have diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & N \arrow[r, "f_1"] \arrow[d, "(1,0)"] & N \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
0 \arrow[r] & N \arrow[r, "{f_2, -f_1}"] \arrow[d, equal] & N^2 \arrow[r, "{(f_1, f_2)}"] \arrow[d, "p_1"] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r, "f_1"] & N \arrow[r] & 0 &
\end{tikzcd}$$

where the vertical maps are (split) exact sequences. The top row is just $K(f_1)$, the bottom row is $K(f_1)[-1]$. A basic result in homological algebra (e.g. [Wei94, Theorem 1.3.1]) asserts that such a short exact sequence of complexes induces a long exact sequence

$$0 \to \mathrm{H}_2(K(f_1, f_2)) \to (0 : f_1)_N \xrightarrow{\delta} (0 : f_1)_N \to \mathrm{H}_1(K(f_1, f_2)) \to N/f_1 \xrightarrow{\delta} N/f_1 \to \mathrm{H}_0(K(f_1, f_2)) \to 0.$$

Since f_1 is a nonzerodivisor, we have $(0 : f_1)_N = 0$. Since f_2 is a nonzerodivisor on N/f_1, and since (by unwinding the long exact sequence above) the map marked δ is multiplication by f_2, we have $\mathrm{H}_1(K(f_1, f_2, N)) = 0$ as well.

A similar argument shows the claim inductively for larger n, see e.g. [Wei94, Corollary 4.5.4].

The statement for the dual Koszul complex holds by Lemma 3.1.9. □

Remark 3.1.13. The Koszul complex is functorial in M in the following way: fix M and $\varphi : M \to A$. Consider an A-linear map $\mu : M' \to M$, and put $\varphi' := \varphi \circ \mu$. Then there is a natural map (of chain complexes)

$$K(\varphi') \to K(\varphi)$$

obtained by observing that $M \mapsto \bigwedge M$ is functorial.

Fixing some $\underline{f}$ as above, we write $\underline{f}^m := (f_1^m, \ldots, f_n^m)$. The functoriality of the Koszul complex as discussed above yields a natural map $K(\underline{f}^{m+1}) \to K(\underline{f}^m)$: the multiplication with $(f_i) : A^n \to A^n$ yields, by passing to exterior powers, a map $\bigwedge(A^n) \to \bigwedge(A^n)$. For example, for $n = 2$, the map reads

$$\begin{tikzcd}
K(\underline{f}^{m+1}) \arrow[d] & A \arrow[r, "{f_2^{m+1}, -f_1^{m+1}}"] \arrow[d, "f_1 f_2"] & A^2 \arrow[r, "{(f_1^{m+1}, f_2^{m+1})}"] \arrow[d, "{\mathrm{diag}(f_1, f_2)}"] & A \arrow[d, "\mathrm{id}"] \\
K(\underline{f}^m) & A \arrow[r, "{f_2^m, -f_1^m}"] & A^2 \arrow[r, "{(f_1^m, f_2^m)}"] & A
\end{tikzcd}$$

Passing to duals, we therefore have maps $K^\vee(\underline{f}^m) \to K^\vee(\underline{f}^{m+1})$.

Definition 3.1.14. We write

$$K^\vee((\underline{f})) := \operatorname{colim}(\ldots \to K^\vee(\underline{f}^m) \to K^\vee(\underline{f}^{m+1}) \to \ldots),$$

where the colimit (indexed by $m \geqslant 0$) is taken in the category of A-modules.

In order to describe this more explicitly, recall that for any A-module N, and $f \in A$, there is an isomorphism (of A-modules)

$$N[f^{-1}] \xrightarrow{\cong} \operatorname{colim}(N \xrightarrow{f} N \xrightarrow{f} N \xrightarrow{f} \dots),$$

under which a fraction $\frac{n}{f^k}$ to n corresponds to n regarded as an element of the k-th copy of N. (If f is a nonzerodivisor on N, i.e., if the multiplication by f is injective, then the colimit above can also be thought of as the union $N \subset \frac{1}{f} \cdot N \subset \frac{1}{f^2} N \subset \dots$.)

Example 3.1.15. If $\underline{f} = (f)$ consists of a single element, then $K^\vee(f^m)$ is the chain complex (located in degrees 0, -1) $A \xrightarrow{f^m} A$, and the transition maps in the above colimit are id in degree 0 and multiplication by f in degree -1. Therefore

$$K^\vee((\underline{f}), N) = \operatorname{colim}(N \xrightarrow{\mathrm{id}} N \xrightarrow{\mathrm{id}} N \to \dots) \to \operatorname{colim}(N \xrightarrow{f} N \xrightarrow{f} N \to \dots) = (N \to N[f^{-1}]).$$

Similarly, for $\underline{f} = (f_1, f_2)$, we get that $K^\vee((\underline{f}), N)$ is the complex

$$N \xrightarrow{(1,1)} N[f_1^{-1}] \times N[f_2^{-1}] \xrightarrow{(1,-1)} N[(f_1 f_2)^{-1}],$$

where the maps are essentially the canonical maps to the localiaztions, with signs as indicated.

Exercises

Exercise 3.1.16. Prove the following converse of Proposition 3.1.11. Let A be a Noetherian local ring, N finitely generated, and $f_1, \dots, f_n \in \mathfrak{m}_A$. Prove that these elements form an N-regular sequence if the Koszul complex $K(f_1, \dots, f_n, N)$ is exact in degrees > 0.

Hint: use the Nakayama lemma (Lemma 1.7.10).

3.2 Definition of Čech cohomology

Notation 3.2.1. Throughout the remainder of this chapter we use the following conventions, unless explicitly stated otherwise:

- *All schemes X are supposed to be quasi-compact and separated.*
- $\mathcal{U}$ denotes a finite covering of X by affine opens:

  $$X = \bigcup_{i=1}^{a} U_i.$$

  For a (finite) subset $I \subset \{1, \dots, a\}$, we write $U_I := \bigcap_{i \in I} U_i$. As X is separated, the schemes U_I are affine (Lemma 2.9.3).
- F denotes a quasi-coherent sheaf on X.

Definition 3.2.2. The *Čech complex* (of F with respect to a fixed covering $\mathcal{U}$) is the cochain complex that in degree $n-1$ is given by $\prod_{I \subset \{1,\dots,a\}, \sharp I = n} F(U_I)$. Thus, in degrees 0 and 1 the complex consists of

$$\prod_{i \leqslant a} F(U_i) \text{ and } \prod_{i<j} F(U_i \cap U_j),$$

respectively. The differential (from degree $n-1$ to degree n) is the map

$$d := d^{n-1} : \prod_{I, \sharp I = n} F(U_I) \to \prod_{I, \sharp I = n+1} F(U_I)$$

which is such that its component $F(U_I) \to F(U_J)$ is zero unless $I \subset J$. If $I \subset J$, i.e., $J = I \cup \{j\}$ for some (unique) $j \leqslant n$, then the map is $(-1)^{k+1}\mathrm{res}^{U_J}_{U_I}$ if j is the k-th element of J. One checks that this is indeed a complex, i.e., that $d^{n+1} \circ d^n = 0$. We denote it by $\Gamma(\mathcal{U}, F)$.

The next step is to construct a complex that is independent of $\mathcal{U}$.

Definition 3.2.3. We put

$$\check{\Gamma}(X, F) := \mathrm{colim}_{\mathcal{U}}\, \Gamma(\mathcal{U}, F),$$

where the colimit runs over *all* affine coverings $\mathcal{U}$; whenever $\mathcal{V}$ is a finer open affine covering than $\mathcal{U}$ (i.e., each subset V_i is contained in some U_j), then the transition map $\Gamma(\mathcal{U}, F) \to \Gamma(\mathcal{V}, F)$ is induced by the restriction maps.

The colimit over the $\mathcal{U}$ above is filtered (given two coverings $\mathcal{U}_1$ and $\mathcal{U}_2$ one can find a covering $\mathcal{U}$ that is finer than both of them, by covering the $U_{1i} \cap U_{2j}$ by smaller open affines).

Definition 3.2.4. Finally, the n-th *Čech cohomology* of F (first with respect to a fixed covering $\mathcal{U}$; then without the choice of such a covering) is defined as

$$\begin{aligned} \mathrm{H}^n_{\mathcal{U}}(X, F) &:= \mathrm{H}^n\Gamma(\mathcal{U}, F) := \ker d^n / \operatorname{im} d^{n-1}, \\ \check{\mathrm{H}}^n(X, F) &:= \mathrm{H}^n\check{\Gamma}(X, F). \end{aligned}$$

It is a generality of homological algebra that H^n commutes with filtered colimits (Exercise 3.2.13), so that

$$\check{\mathrm{H}}^n(X, F) = \mathrm{colim}_{\mathcal{U}}\, \mathrm{H}^n\Gamma(\mathcal{U}, F).$$

Example 3.2.5. Consider $X = \mathbf{A}^2 \backslash \{0\}$, endowed with the covering by $U_1 = \mathbf{A}^1 \times \mathbf{G}_{\mathrm{m}} = \operatorname{Spec} \mathbf{Z}[t_1, t_2^{\pm 1}]$ and $U_2 = \mathbf{G}_{\mathrm{m}} \times \mathbf{A}^1 = \operatorname{Spec} \mathbf{Z}[t_1^{\pm 1}, t_2]$. We have $U_1 \cap U_2 = \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}$, and therefore the Čech complex for this covering reads

$$\Gamma_{\mathcal{U}}(X, \mathcal{O}_X) = \mathbf{Z}[t_1, t_2^{\pm 1}] \times \mathbf{Z}[t_1^{\pm 1}, t_2] \stackrel{d}{\to} \mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}]. \tag{3.2.6}$$

(The groups are located in degree 0 and 1, respectively). The differential d is given by $d(f_1, f_2) = f_2 - f_1$. We have

$$\mathrm{H}^0_{\mathcal{U}}(X, \mathcal{O}_X) = \ker d = \mathcal{O}_X(X) = \mathbf{Z}[t_1, t_2].$$

Indeed, this holds by the sheaf property of $\mathcal{O}_X$ (cf. also Example 2.1.2). Moreover,

$$\mathrm{H}^1_{\mathcal{U}}(X, \mathcal{O}_X) = \operatorname{coker} d \cong \bigoplus_{n_1, n_2 < 0} \mathbf{Z} t_1^{n_1} t_2^{n_2}.$$

All other cohomology groups vanish, since the complex $\Gamma_{\mathcal{U}}(X, \mathcal{O}_X)$ is zero there. We will shortly relate this computation to the cohomology groups $\check{\mathrm{H}}^n(\mathbf{P}^1, \bigoplus_{e \in \mathbf{Z}} \mathcal{O}(e))$ (Theorem 3.4.1). We note that both U_i are affine; we will shortly prove that this implies $\check{\mathrm{H}}^*(X, \mathcal{O}_X) = \check{\mathrm{H}}^*_{\mathcal{U}}(X, \mathcal{O}_X)$ (Proposition 3.3.4).

By the sheaf property, we have (for any X, F and $\mathcal{U}$) an isomorphism

$$F(X) \stackrel{\cong}{\to} \mathrm{H}^0_{\mathcal{U}}(X, F) (= \ker d^0). \tag{3.2.7}$$

In particular, this does not depend on $\mathcal{U}$. Of course, then taking the colimit over all affine coverings $\mathcal{U}$ does not do anything, so

$$F(X) \cong \check{\mathrm{H}}^0(X, F).$$

The following fact is the workhorse when it comes to computing cohomology groups in practice. Recall from Lemma 2.10.16 that $\mathrm{QCoh}(X)$ is an abelian category, so we can consider exact sequences

$$0 \to F' \to F \to F'' \to 0$$

in this category. Concretely, given F', F, F'' and maps of $\mathcal{O}_X$-modules $F' \to F$ and $F \to F''$ (whose composition is zero), a sequence is exact if for any open *affine* $U \subset X$ belonging to some fixed covering $\mathcal{U}$, the sequence

$$0 \to F'(U) \to F(U) \to F''(U) \to 0 \tag{3.2.8}$$

is exact. Note that this is *not* implying that

$$0 \to F'(X) \to F(X) \to F''(X) \to 0$$

is an exact sequence. The next lemma does state that it is exact except possibly for the surjectivity of the map $F(X) \to F''(X)$. This map is, in general, not surjective, and the failure to be surjective is measured by $\check{\mathrm{H}}^1(X, F')$.

Lemma 3.2.9. For an exact sequence as above, there is a long exact sequence as follows, where $\check{\mathrm{H}}^n(F) := \check{\mathrm{H}}^n(X, F)$ etc.

$$0 \to \check{\mathrm{H}}^0(F') \to \check{\mathrm{H}}^0(F) \to \check{\mathrm{H}}^0(F'') \to \check{\mathrm{H}}^1(F') \to \check{\mathrm{H}}^1(F) \to \check{\mathrm{H}}^1(F'') \to \dots \tag{3.2.10}$$

In particular, the map

$$\check{\mathrm{H}}^0(F) \to \check{\mathrm{H}}^0(F'')$$

is surjective if and only if $\check{\mathrm{H}}^1(F') \to \check{\mathrm{H}}^1(F)$ is injective (and this is the case if $\check{\mathrm{H}}^1(F') = 0$, but not necessarily in general).

Proof. Fix an affine covering $\mathcal{U}$. Applying the exact sequence (3.2.8) to the multiple intersections U_I (which are all affine) we obtain an exact sequence of complexes

$$0 \to \Gamma(\mathcal{U}, F') \to \Gamma(\mathcal{U}, F) \to \Gamma(\mathcal{U}, F'') \to 0$$

(i.e., the terms of these complexes form exact sequences of abelian groups). The assertion then is nothing but the long exact sequence of cohomology groups, e.g. [Wei94, Theorem 1.3.1]. ☐

Example 3.2.11. Consider $X = \mathbf{P}^1 = \operatorname{Proj} \mathbf{Z}[t_0, t_1]$ and $i : Y := V(t_0) \to X$. Note that $Y = \operatorname{Proj} \mathbf{Z}[t_0, t_1]/t_0 = \operatorname{Proj} \mathbf{Z}[t_1] = \mathbf{P}^0 = \operatorname{Spec} \mathbf{Z}$.

There is an exact sequence of graded $\mathbf{Z}[t_0, t_1]$-modules, where (-2) etc. denotes the Serre twist (Definition 2.10.14):

$$0 \to \mathbf{Z}[t_0, t_1](-2) \xrightarrow{t_0} \mathbf{Z}[t_0, t_1](-1) \to \mathbf{Z}[t_1](-1) \to 0.$$

We can apply the functor $\widetilde{-}$ (Definition and Lemma 2.10.13; note that by its very definition this is an exact functor), and obtain an exact sequence

$$0 \to \mathcal{O}_X(-2) \xrightarrow{t_0} \mathcal{O}_X(-1) \to i_* \mathcal{O}_{\operatorname{Spec} \mathbf{Z}} \to 0 \tag{3.2.12}$$

in $\mathrm{QCoh}(X)$. (Initially, it would be more appropriate to write $i_* \mathcal{O}_{\operatorname{Spec} \mathbf{Z}}(-1)$ at the right; however note that $\mathbf{P}^0 = \operatorname{Proj} \mathbf{Z}[t_1]$ is isomorphic to $D_+(t_1)$, on which $\mathcal{O}(-1) \cong \mathcal{O}$). We inspect the above exact sequence (3.2.10), using Lemma 2.10.15 for the two left hand groups:

$$0 \to \underbrace{\Gamma(\mathcal{O}(-2))}_{=0} \to \underbrace{\Gamma(\mathcal{O}(-1))}_{=0} \to \Gamma(i_* \mathcal{O}) = \mathbf{Z} \to \check{\mathrm{H}}^1(X, \mathcal{O}(-2)) \to \check{\mathrm{H}}^1(X, \mathcal{O}(-1)) \to \dots$$

We observe that the global sections of $\mathcal{O}(-1) \to i_* \mathcal{O}$ are *not* surjective. In Theorem 3.4.1, we will compute the next groups in the long exact sequence to be

$$\check{\mathrm{H}}^1(X, \mathcal{O}_X(-2)) = \mathbf{Z} \to \check{\mathrm{H}}^1(X, \mathcal{O}_X(-1)) = 0.$$

So the failure of the surjectivity on global sections for $\mathcal{O}(-1) \to i_* \mathcal{O}$ is captured by the non-vanishing of this first cohomology group.

Exercises

Exercise 3.2.13. Let I be a filtered category (a useful example to keep in mind is $I = \{0 \to 1 \to 2 \to \dots\}$). Let $C : I \to \mathrm{Ch}$ be a functor, i.e., for each $i \in I$ there is a chain complex C_i, and whenever $i \to j$, there is a map (of chain complexes) $C_i \to C_j$. Let $C_\infty := \operatorname{colim} C_i$ be the (filtered) colimit of these chain complexes. Establish an isomorphism

$$\operatorname{colim} \mathrm{H}^n(C_i) \stackrel{\cong}{\to} \mathrm{H}^n(C_\infty).$$

Hint: first prove a similar claim for the kernel of the differential, and its image.

Exercise 3.2.14. Let X be quasi-compact and separated, and $F_i \in \mathrm{QCoh}(X)$, where $i \in I$ for some index set I.

(1) Show that the presheaf $U \mapsto \bigoplus_i F_i(U)$ is in fact a quasi-coherent sheaf. We denote it by $\bigoplus_i F_i$. Prove that for any $G \in \mathrm{QCoh}(X)$, there is a natural isomorphism $\mathrm{Hom}_{\mathrm{QCoh}(X)}(\bigoplus_i F_i, G) = \prod_i \mathrm{Hom}_{\mathrm{QCoh}(X)}(F_i, G)$ (so this is indeed the coproduct in the category $\mathrm{QCoh}(X)$).

(2) Prove that cohomology (on a quasi-compact separated scheme) commutes with direct sums, i.e., establish an isomorphism

$$\bigoplus_i \check{\mathrm{H}}^*(X, F_i) \stackrel{\cong}{\to} \check{\mathrm{H}}^*(X, \bigoplus_i F_i).$$

Exercise 3.2.15. For a scheme X/k, the *Euler characteristic* of $F \in \mathrm{QCoh}(X)$ is defined to be

$$\chi(F) := \chi(X, F) := \sum_{s \geqslant 0} (-1)^s \dim_k \mathrm{H}^s(X, F),$$

provided that each dimension in this alternating sum is finite, and provided that only finitely many groups are nonzero.

(1) For a short exact sequence

$$0 \to F' \to F \to F'' \to 0$$

prove that if χ is defined for two out of the three sheaves, then it is also defined for the third one, and that in this event the formula

$$\chi(F') + \chi(F'') = \chi(F) \qquad (3.2.16)$$

holds.

(2) Deduce that for an exact sequence

$$0 \to F_n \to F_{n-1} \to \dots F_0 \to 0$$

one has

$$\chi(F_0) = \sum_{q > 0} (-1)^q \chi(X, F_q).$$

Hint: establish exact sequences $0 \to \ker(F_q \to F_{q-1}) \to F_q \to \operatorname{coker}(F_{q+1} \to F_q) \to 0$.

3.3 Čech cohomology of affine schemes

Given (3.2.7), our interest is now to understand higher cohomology groups. Here is a first such computation. The exposition below follows [Gro61, §III.2].

Lemma 3.3.1. Let $X = \operatorname{Spec} A$, $F \in \mathrm{QCoh}(X)$ and consider a finite covering $\mathcal{U}$ of X by basic open subsets $U_i = D(f_i)$, $i \leqslant a$. Then

$$\mathrm{H}^n_{\mathcal{U}}(X, F) = 0 \text{ for } n \geqslant 1.$$

Proof. Let $N := F(X)$ (so that $\widetilde{N} = F$). For $m \geqslant 0$, write $\underline{f}^m = (f_1^m, \dots, f_a^m)$.

Since $X = \bigcup D(f_i) = \bigcup D(f_i^m)$, we have that 1 is a linear combination of the f_i^m (Lemma 1.1.10(2)). Thus multiplication by 1, i.e., the identity map on the dual Koszul complex $K^\vee(\underline{f}^m, N)$ is null-homotopic. Hence this is an exact complex (Corollary 3.1.8). However, this complex identifies with the following (with the nonzero terms being in degrees 0, 1, 2 etc.)

$$0 \to N \to \prod_{i_1} N[f_{i_1}^{-1}] \to \prod_{i_1 < i_2} N[(f_{i_1} f_{i_2})^{-1}] \to \dots \tag{3.3.2}$$

This is just the complex $N \to \Gamma_{\mathcal{U}}(X, \widetilde{N})$, so we are done. □

Corollary 3.3.3. If $X = \operatorname{Spec} A$, $F = \widetilde{N}$ (for $N \in \operatorname{Mod}_A$), we have

$$\check{\mathrm{H}}^n(X, F) = \begin{cases} N & n = 0 \\ 0 & n > 0 \end{cases}$$

Proof. Any affine covering $\mathcal{U}$ of X admits a refinement by a covering consisting of basic open subsets. When computing $\check{\mathrm{H}}^n(X, F) = \operatorname{colim}_{\mathcal{U}} \mathrm{H}^n_{\mathcal{U}}(X, F)$, it is enough to take the colimit over those coverings. The groups at the right are, however, $F(X)$ for $n = 0$ and 0 for $n \geqslant 1$ by Lemma 3.3.1. □

We now prove that Cech cohomology can be computed using any affine covering.

Proposition 3.3.4. Let X be a quasi-compact and separated (but not necessarily affine) scheme, and $\mathcal{U}$ a fixed covering by affines. Then, for any $F \in \operatorname{QCoh}(X)$ there is an isomorphism

$$\mathrm{H}^n_{\mathcal{U}}(X, F) \xrightarrow{\cong} \check{\mathrm{H}}^n(X, F).$$

Proof. Given that the colimit $\operatorname{colim}_{\mathcal{U}} \mathrm{H}^n \Gamma_{\mathcal{U}}(X, F)$ is filtered, it is enough to show for any open affine covering $\mathcal{V} : X = \bigcup V_j$ that is finer than $\mathcal{U}$, we have a quasi-isomorphism

$$\Gamma_{\mathcal{U}}(X, F) \to \Gamma_{\mathcal{V}}(X, F).$$

By considering the covering "$\mathcal{U} \cup \mathcal{V}$": $X = \bigcup U_i \cup \bigcup V_j$, we may assume the subsets of $\mathcal{U}$ are part of the subsets of $\mathcal{V}$. By an induction it therefore suffices to show that we have a quasi-isomorphism as above in the situation where $\mathcal{V}$ is of the form $X = \bigcup_i U_i \cup U_0$, for some arbitrary open affine subset $U_0 \subset X$.

We have an exact sequence of chain complexes (i.e., each column below is exact)

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \dots & 0 \arrow[d] & 0 \arrow[d] & \\
F(U_0) \arrow[r] \arrow[d] & \prod_i F(U_0 \cap U_i) \arrow[r] \arrow[d] & \dots \arrow[r] & \prod'_{\sharp I = n} F(U_I) \arrow[r] \arrow[d] & \prod'_{\sharp I = n+1} F(U_I) \arrow[r] \arrow[d] & \dots \\
\prod_{i \geqslant 0} F(U_i) \arrow[r] \arrow[d] & \prod_{\sharp I = 2} F(U_I) \arrow[r] \arrow[d] & \dots \arrow[r] & \prod_{\sharp I = n} F(U_I) \arrow[r] \arrow[d] & \prod_{\sharp I = n+1} F(U_I) \arrow[r] \arrow[d] & \dots \\
\prod_{i \neq 0} F(U_i) \arrow[r] \arrow[d] & \prod''_{\sharp I = 2} F(U_I) \arrow[r] \arrow[d] & \dots \arrow[r] & \prod''_{\sharp I = n} F(U_I) \arrow[r] \arrow[d] & \prod''_{\sharp I = n+1} F(U_I) \arrow[r] \arrow[d] & \dots \\
0 & 0 & \dots & 0 & 0 &
\end{tikzcd}$$

Here the notation $\prod'$ means the product over the index sets such that $0 \in I$, while $\prod''$ means $0 \notin I$. (The leftmost column is in cochain degree 1, the right most in degree $n+1$.) The middle row is just $\Gamma_{\mathcal{V}}(X, F)$, while the bottom row is $\Gamma_{\mathcal{U}}(X, F)$. The top row is comprised of $F(U_0)$ in degree 1, and in higher degrees of $\Gamma_{U_0 \cap \mathcal{U}}(U_0, F)$, where $U_0 \cap \mathcal{U}$ is the covering of U_0 given by $U_0 = \bigcup_i U_0 \cap U_i$.

By a general fact in homological algebra [Wei94, Theorem 1.3.1], the map from middle to bottom row is a quasi-isomorphism if (and only if) the top row is an exact complex.

In other words, we have to see that $\Gamma_{\mathcal{U} \cap U_0}$ is exact in degrees $\geqslant 1$. In other words, we have reduced our claim for X to the one for U_0, i.e., we may henceforth assume X is affine.

We consider $X = \bigcup U_i$ as before. Our goal is to prove that

$$0 \to F(X) \xrightarrow{\text{res}} \underbrace{\prod_i F(U_i) \to \prod_{i_1 < i_2} F(U_{i_1, i_2}) \to \dots}_{\Gamma_{\mathcal{U}}(X,F)} \tag{3.3.5}$$

is an exact sequence. We first do this in the special case where one of the open subsets, say U_0 is equal to X.

In this case we observe that there is a commutative diagram of chain complexes as below, where the middle row is the complex (3.3.5) above, and where each column is a (split) short exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \prod_{\sharp I=1, 0 \in I} F(U_I) \arrow[r] \arrow[d] & \prod_{\sharp I=2, 0 \in I} F(U_I) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & F(X) \arrow[r, "\text{res}"] \arrow[d] & \prod_{\sharp I=1} F(U_I) \arrow[r] \arrow[d] & \prod_{\sharp I=2} F(U_I) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & F(X) \arrow[r, "\text{res}"] & \prod_{\sharp I=1, 0 \notin I} F(U_I) \arrow[r] & \prod_{\sharp I=2, 0 \notin I} F(U_I) \arrow[r] & \dots
\end{tikzcd}$$

Writing T, M, B for top, middle and bottom row, we observe that $T = B[-1]$, so that $\mathrm{H}^n(T) = \mathrm{H}^{n-1}(B)$, and the long exact cohomology sequence associated to this short exact sequence of complexes then reads

$$\mathrm{H}^n(T) \to \mathrm{H}^n(M) \to \mathrm{H}^n(B) \xrightarrow{\delta} \mathrm{H}^{n+1}(T) = \mathrm{H}^n(B)$$

and one checks that the map δ is the identity (for all n). Therefore $\mathrm{H}^n(M) = 0$, i.e., M is exact.

We now prove that the exactness of the complex in (3.3.5) in the general case where $X = \operatorname{Spec} A$ is affine. We can pick a refinement of the covering $\mathcal{U}$ that consists of open subsets $V = D(f)$ (i.e., each $D(f)$ is contained in some U_i). It suffices to prove the exactness of the complex after localizing at any such f. Since F is quasi-coherent we have

$$0 \to F(X)[f^{-1}] = F(D(f)) \to \prod_i F(U_i)[f^{-1}] = \prod_i F(U_i \cap D(f)) \to \dots$$

In other words, this localization is the complex (3.3.5), but with X being replaced by $D(f)$ and the covering $\mathcal{U}$ being replaced by its intersection with $D(f)$. Since $D(f)$ is contained in some U_i, the exactness in this case holds by the case handled previously. $\square$

Exercises

Exercise 3.3.6. Consider a pullback diagram

$$\begin{tikzcd}
X' \arrow[r, "s'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' = \operatorname{Spec} A' \arrow[r, "s"] & S = \operatorname{Spec} A
\end{tikzcd}$$

where X is quasi-compact and separated, and the bottom map s is induced by a flat map $A \to A'$. (A typical example is when A is a field, in which case any A' is flat.) Prove that there is a natural isomorphism

$$A' \otimes_A \mathrm{H}^n(X, F) \xrightarrow{\cong} \mathrm{H}^n(X', s'^* F)$$

for any $F \in \mathrm{QCoh}(X)$. One refers to this by saying that *cohomology commutes with flat base change*.

Exercise 3.3.7. Let X be a quasi-compact and separated scheme that admits a covering by n affine open subschemes. Prove that $\mathrm{H}^k(X, F) = 0$ for $k \geqslant n$ for any $F \in \mathrm{QCoh}(X)$.

3.4 Čech cohomology of projective space

We now compute the cohomology of the line bundles $\mathcal{O}(e)$ on projective space $X := \mathbf{P}^n_B$ (for some ring B). Recall that $X = \bigcup_{i=0}^n D_+(t_i)$, and each $D_+(t_i)$ is isomorphic to $\mathbf{A}^n_B$. In particular X is quasi-compact. It is also separated (Example 2.9.6). We are thus in a position to compute Cech cohomology to begin with (cf. Notation 3.2.1), which we will do using the above affine open covering $\mathcal{U}$.

Instead of focussing on a single $\mathcal{O}(e)$ it will be convenient to consider

$$\mathcal{O}_X(\star) := \bigoplus_{e \in \mathbf{Z}} \mathcal{O}_X(e) (\in \mathrm{QCoh}(X)).$$

By Exercise 3.2.14, we have

$$\mathrm{H}^r(X, \mathcal{O}_X(\star)) = \mathrm{H}^r(X, \bigoplus_{e \in \mathbf{Z}} \mathcal{O}_X(e)) = \bigoplus_{e \in \mathbf{Z}} \mathrm{H}^n(X, \mathcal{O}_X(e))$$

so that the cohomology groups of $\mathcal{O}_X(\star)$ are $\mathbf{Z}$-graded. We write $A = B[t_0, \dots, t_n]$, which is graded such that $\deg t_i = 1$ (Example 2.3.3).

Theorem 3.4.1. With the above notation, we have isomorphisms of graded A-modules as follows:

- $\mathrm{H}^0(X, \mathcal{O}_X(\star)) \cong A$ (so that $\mathrm{H}^0(X, \mathcal{O}_X(e)) = A_e$, the degree e component of A; note this vanishes if $e < 0$)
- $\mathrm{H}^r(X, \mathcal{O}_X(\star)) = 0$ for $r \neq 0, n+1$,
- $\mathrm{H}^n(X, \mathcal{O}_X(\star)) = A[(t_0 \dots t_n)^{-1}]/A[(t_0 \dots t_n)^{-1}]_{\geqslant 0} = \bigoplus_{\underline{a} > 0} B t_0^{-a_0} \dots t_n^{-a_n}$, i.e., a free B-module with a basis given by monomials as indicated, where the multiindex $\underline{a} = (a_0, \dots, a_n) \in \mathbf{Z}^{n+1}$ is such that $a_i > 0$ for all i. (The subscript "$\geqslant 0$" indicates the A-submodule spanned by monomials in which at least one t_i appears with a non-negative exponent.) In particular, butting $b_i := -(a_1 - 1)$ in the above expressions the summands for individual Serre twists e have the following descriptions (i.e., there are isomorphisms of B-modules)

$$\begin{aligned} \mathrm{H}^n(X, \mathcal{O}_X(e)) &= \bigoplus_{b_i \leqslant 0, \sum b_i = e + (n+1)} B t_0^{b_0} \cdots t_n^{b_n}, \qquad (3.4.2) \\ \mathrm{H}^n(X, \mathcal{O}_X(e)) &= 0 \text{ for } e > -n-1 \\ \mathrm{H}^n(X, \mathcal{O}_X(-n-1)) &= B. \end{aligned}$$

Proof. The proof consists in essentially expressing the complex $\Gamma_{\mathcal{U}}(X, \mathcal{O}(\star))$ in terms of an appropriate Koszul complex.

Consider the sequence $\underline{t} := (t_0, \dots, t_n)$ in A, and $\underline{t}^m := (t_0^m, \dots, t_n^m)$. The dual Koszul complex (Definition 3.1.14)

$$K^\vee((\underline{t})) = \mathrm{colim}_{m \geqslant 1} K^\vee(\underline{t}^m)$$

reads as follows (compare with (3.3.2); the term $B[t_\bullet]$ is in cochain degree 0 where $t_\bullet$ serves as a shorthand for "$t_0, \dots, t_n$"; the rightmost term is in cochain degree $n+1$):

$$\begin{tikzcd} B[t_\bullet] \arrow[r] & \prod_i B[t_\bullet, t_i^{-1}] \arrow[r] \arrow[d, equal] & \prod_{i_1<i_2} B[t_\bullet, (t_{i_1}t_{i_2})^{-1}] \arrow[r] \arrow[d, equal] & \dots \arrow[r] & B[t_\bullet, (t_0t_1\dots t_n)^{-1}] \arrow[d, equal] \\ & \prod_i \Gamma(U_i, \mathcal{O}(\star)) \arrow[r] & \prod_{i_1<i_2} \Gamma(U_{i_1} \cap U_{i_2}, \mathcal{O}(\star)) \arrow[r] & \dots \arrow[r] & \Gamma(\bigcap_{i\leqslant n} U_i, \mathcal{O}_X(\star)) \end{tikzcd}$$

Underneath, we have indicated the agreement of the degree $\geqslant 1$-parts of $K^\vee((\underline{t}))$ with the Cech complex $\Gamma_{\mathcal{U}}(X, \mathcal{O}_X(\star))$. (An inspection of the definition shows the signs of the differentials agree as well.)

For $m \geqslant 1$, the sequence $\underline{t}^m$ is a regular sequence in A, so that Proposition 3.1.11 gives

$$\mathrm{H}^k(K^\vee(\underline{t}^m)) = \begin{cases} A/(t_i^m) & k = n+1 \\ 0 & k \neq n+1 \end{cases}$$

The nonzero group is a free B-module of finite rank spanned by monomials of the form $t^{m-\underline{a}} := \prod_{i=0}^n t_i^{m-a_i}$, where $0 < a_i \leqslant m$. The (filtered) colimit over $m \geqslant 1$ is formed using the transition map $K^\vee(\underline{t}^m) \to K^\vee(\underline{t}^{m+1})$ that is given by multiplication with $t_0t_1\dots t_n$. It maps $t^{m-\underline{a}}$ to $t_{(m+1)-\underline{a}}$, so that passing to the colimit gives

$$\mathrm{H}^k(K^\vee((\underline{t}))) = \operatorname{colim}_m \mathrm{H}^k(K^\vee(\underline{t}^m)) = \begin{cases} \bigoplus_{\underline{a}>0} Bt_\bullet^{-\underline{a}} & k = n+1 \\ 0 & k \neq n+1 \end{cases}$$

Here the direct sum is indexed by all multi-indices $\underline{a} = (a_0, \dots, a_n)$ with $a_i > 0$ for all i, as claimed above

In other words, the group $\mathrm{H}^0_{\mathcal{U}}(X, \mathcal{O}_X(\star))$, which is the kernel of the leftmost differential in the bottom complex, identifies with $B[t_\bullet]$. (This was already clear from (3.2.7) above.) For $r \geqslant 1$, we have

$$\check{\mathrm{H}}^r(X, \mathcal{O}_X(\star)) = \mathrm{H}^r_{\mathcal{U}}(X, \mathcal{O}_X(\star)) = \mathrm{H}^{r+1}(K^\vee((\underline{t}))).$$

It remains to observe that the above identification of the Cech complex with the Koszul complex respects the $\mathbf{Z}$-gradings;

- the Koszul complex for the $\mathbf{Z}$-*graded* A-module $A^{n+1}(-1)$ (and the graded map $A^{n+1}(-1) \xrightarrow{t_i} A$) is $\mathbf{Z}$-graded as well. Concretely, this is simply the $\mathbf{Z}$-grading on (the localizations of) $B[t_\bullet]$ where $\deg t_i = 1$.
- The Cech complex is $\mathbf{Z}$-graded by means of the grading on $\mathcal{O}_X(\star)$.

This finishes the computation of the cohomology groups. □

Outlook 3.4.3. Consider the following maps

$$\begin{tikzcd} U := \mathbf{A}^{n+1}_B \backslash \{0\} \arrow[r, hook, "j"] \arrow[d, "q"] & \mathbf{A}^{n+1}_B \arrow[d, "a"] \\ \mathbf{P}^n_B = U/\mathbf{G}_\mathrm{m} \arrow[r, "\pi"] & \operatorname{Spec} B. \end{tikzcd}$$

There is a formalism of so-called *derived categories* of quasi-coherent sheaves on these schemes, and, for any map of schemes $f : X \to Y$, the *derived functor* $Rf_* : \mathrm{D}_{\mathrm{QCoh}}(X) \to \mathrm{D}_{\mathrm{QCoh}}(Y)$. Then the Čech cohomology groups $\mathrm{H}^*(\mathbf{P}^n, \mathcal{O})$ are the cohomology groups of the complex $R\pi_*\mathcal{O}_{\mathbf{P}^n} \in \mathrm{D}_{\mathrm{QCoh}}(\operatorname{Spec} \mathbf{Z}) = \mathrm{D}(\mathrm{Mod}_{\mathbf{Z}})$; similarly $\mathrm{H}^*(\mathbf{P}^n, \bigoplus_e \mathcal{O}(e))$ is related to $R\pi_* \bigoplus_e \mathcal{O}(e)$. The above computation of these cohomology groups are explained by observing that

$$\bigoplus_e \mathcal{O}(e) = Rq_*\mathcal{O}_U,$$

which follows essentially from the fact that q is affine (!), which by Lemma 3.4.4 implies that $Rq_* \mathcal{O}_U = q_* \mathcal{O}_U = \bigoplus_e \mathcal{O}(e)$. This implies

$$R\pi_* \bigoplus_e \mathcal{O}(e) = R\pi_* Rq_* \mathcal{O}_U = Ra_* Rj_* \mathcal{O}_U.$$

Since $\mathbf{A}^{n+1}$ is affine, the vanishing of higher cohomology of quasi-coherent sheaves means that Ra_* is essentially just a forgetful functor. By contrast, the inclusion j is *not* affine, and $Rj_* \mathcal{O}_U$ is given by the Cech comples $\Gamma(\mathcal{U}, \mathcal{O}_U)$, for the standard covering of $\mathbf{A}^{n+1} \backslash 0$ by $n+1$ copies of $\mathbf{G}_{\mathrm{m}} \times \mathbf{A}^n$, cf. (3.2.6).

Lemma 3.4.4. If $f : X \to Y$ is an *affine* morphism (Definition 2.5.1, for example f could be finite or, more specifically, a closed immersion), then for any $F \in \mathrm{QCoh}(X)$ there is a natural isomorphism

$$\check{\mathrm{H}}^*(Y, f_* F) \stackrel{\cong}{\to} \check{\mathrm{H}}^*(X, F).$$

Proof. Fix an affine open covering $\mathcal{U}$ of Y. By assumption, the open covering $f^{-1}(\mathcal{U})$ of X consists of *affine* opens. Using Proposition 3.3.4, we can compute Cech cohomology using these covers. By definition, $\Gamma(U_I, f_* F) = \Gamma(f^{-1}(U_I), F)$, so the Cech complexes are isomorphic, hence so are their cohomology groups. $\square$

Theorem 3.4.5. (*Bézout's theorem*) Let $f_i \in k[t_0, \dots, t_n]$, $1 \leqslant i \leqslant n$ be homogeneous polynomials of degree d_i. Consider the *hypersurface*

$$H_i := V(f_i) := \mathrm{Proj}\, k[t_0, \dots, t_n]/f_i \subset \mathbf{P}^n_k =: X.$$

We suppose that the scheme-theoretic intersection (Exercise 2.6.5)

$$Y := H_1 \cap \dots \cap H_n = V(f_1, \dots, f_n)$$

is finite over k. Then

$$\dim_k \mathcal{O}_Y(Y) = \prod_i d_i.$$

In particular, if k is algebraically closed and Y is reduced, then

$$\sharp Y = \prod_i d_i.$$

Proof. Let $E := \bigoplus_i \mathcal{O}(-d_i)$. Note this is a locally free sheaf of rank n on $\mathbf{P}^n_k$. We consider the Koszul complex for $E \to \mathcal{O}$ given by multiplication with the f_i. (This is possible since the definition of the Koszul complex can be adapted to any quasi-coherent sheaf E, and map $E \to \mathcal{O}_X$; equivalently, but more closely related to the above computation, one may also consider the graded $A := k[t_0, \dots, t_n]$-module $M := \bigoplus_i A(-d_i)$ and form the Koszul complex of this graded A-module with respect to the map $M \to A$ given by multiplication with the f_i; then apply the (exact) functor $\widetilde{-}$ in Definition and Lemma 2.10.13.) As in Example 3.2.11, we obtain an exact sequence (in $\mathrm{QCoh}(\mathbf{P}^n)$)

$$0 \to \bigwedge^n E \to \dots \to E \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0.$$

where $i : Y \to \mathbf{P}^n_k$ is the closed embedding. For any $F = \bigwedge^a E$ appearing in the complex above, the *Euler characteristic* $\chi(F)$ (cf. Exercise 3.2.15) is defined. Indeed, $\chi(\mathcal{O}(-d_i))$ is defined (all but possibly two cohomology groups of $\mathcal{O}_X(-d_i)$ are zero). Therefore the Euler characteristic is also well-defined for E and its exterior powers, as one checks inductively using that

$$\bigwedge^r (\mathcal{O}(d_0) \oplus E') = \left(\bigwedge^0 \mathcal{O}(d_0) \otimes \bigwedge^r E' \right) \oplus \left(\bigwedge^1 \mathcal{O}(d_0) \oplus \bigwedge^{r-1} E' \right) = \bigwedge^r E' \oplus (\bigwedge^{r-1} E')(d_0).$$

We obtain

$$\chi(i_*\mathcal{O}_Y) = \sum_q (-1)^q \chi\left(\bigwedge^q E\right).$$

Crucially, the right hand side depends only on d_i, but not on the f_i. To compute $\chi(i_*\mathcal{O}_Y)$, we may therefore assume $f_i = t_i^{d_i}$. Similarly to Example 3.2.11, we have $Y := \bigcap_{i=1}^n V(t_i^{d_i}) = \operatorname{Spec} k[x_1, \dots, x_n]/(x_i^{d_i})$ (noting that $Y \subset D_+(t_0) = \operatorname{Spec} k[\frac{t_i}{t_0}] =: \operatorname{Spec} k[x_1, \dots, x_n]$). Using Lemma 3.4.4 and Corollary 3.3.3, we get

$$\chi(X, i_*\mathcal{O}_Y) = \chi(Y, \mathcal{O}_Y) = \dim_k \mathcal{O}_Y(Y) = \dim_k k[x_i]/x_i^{d_i} = \prod_i d_i.$$

The additional claim holds since for any finite k-algebra A (such as $\mathcal{O}_Y(Y)$) there is an isomorphism $A = \bigoplus_{\mathfrak{m} \subset A} A_{\mathfrak{m}}$ (cf. Exercise 1.7.29). If A is reduced, then $A_{\mathfrak{m}}$ is a domain, which is therefore a field extension of k. If $k = \overline{k}$, $A_{\mathfrak{m}} = k$. □

3.5 Finiteness of cohomology

One consequence of Theorem 3.4.1 is that the cohomology groups of $\mathcal{O}(d)$ on $\mathbf{P}^n$ are finitely generated. In this section, we provide a more general statement asserting such finiteness results. We begin by discussing the necessary finiteness condition on quasicoherent sheaves F.

3.5.1 Coherent sheaves

Let A be a Noetherian ring. Recall that for an A-module M, the following are equivalent:

(1) M is finitely generated, i.e., lies in an exact sequence

$$A^n \to M \to 0.$$

(2) M is finitely presented, i.e., lies in an exact sequence

$$A^m \to A^n \to M \to 0.$$

(3) M is *coherent*, i.e. M is finitely generated and the kernel of any (not necessarily surjective) map $A^n \to M$ is finitely generated.

Indeed, the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious, and for (1) $\Rightarrow$ (3), the kernel of $A^n \to M$ is a submodule of a finitely generated A-module, and for Noetherian rings, these are finitely generated.

Remark 3.5.1. For simplicity, we only consider coherent modules over Noetherian rings. A well-behaved theory of coherent modules exists for a more general class of rings called *coherent rings*, cf. [Stacks, Tag 05CV] onwards. A ring is coherent if it satisfies condition (3) above, i.e., if any finitely generated ideal is finitely presented. Any Noetherian ring is coherent, but not conversely. An example of a coherent, non-Noetherian ring is the ring $\mathcal{O}(Z)$ of holomorphic functions on a polydisk $Z := \{(z_1, \dots, z_n) \in \mathbf{C}^n, |z_i| \leqslant 1\}$.

Throughout this section, let X be a locally Noetherian scheme (Definition 2.1.10), so $X = \bigcup \operatorname{Spec} A_i$, with A_i being Noetherian. A typical case is if X is locally of finite type over $\operatorname{Spec} A$, for a Noetherian ring A, e.g. a field or $\mathbf{Z}$. We are going to introduce the concept of a coherent sheaf, which extends the above notion of finitely generated modules.

Definition 3.5.2. A quasi-coherent sheaf F on a locally Noetherian scheme X is called *coherent* if for any open affine $U = \operatorname{Spec} A \subset X$, $F(U)$ is a coherent (equivalently, finitely generated) A-module. Coherent sheaves spann a full subcategory $\operatorname{Coh}(X) \subset \operatorname{QCoh}(X)$.

Example 3.5.3.
- The structural sheaf $\mathcal{O}_X$ is a coherent sheaf.
- More generally, for a closed immersion $i : Y \to X$, $i_*\mathcal{O}_Y \in \mathrm{Coh}(X)$. Indeed, locally on X, i is of the form $\operatorname{Spec} A/I \to \operatorname{Spec} A$.
- Even more generally, if $f : Y \to X$ is a finite morphism, and $F \in \mathrm{Coh}(Y)$, then $f_*F \in \mathrm{Coh}(X)$: locally on X, f is given by $\operatorname{Spec} A \to \operatorname{Spec} B$ for A being finite (as a module!) over B, and so any finitely generated A-module is also finitely generated when regarded as a B-module.
- Lemma 2.10.15 shows that for the structural map $f : \mathbf{P}^n \to \operatorname{Spec} \mathbf{Z}$,
$$f_*\mathcal{O}(e) \in \mathrm{Coh}(\operatorname{Spec} \mathbf{Z}).$$
- However, for $f : \mathbf{A}^1 \to \operatorname{Spec} \mathbf{Z}$,
$$f_*\mathcal{O}_{\mathbf{A}^1} \notin \mathrm{Coh}(\operatorname{Spec} \mathbf{Z})$$
since its global sections are $\mathbf{Z}[t]$, which is not finitely generated (as a $\mathbf{Z}$-module).
- Another notable permanence property is that for *any* map $f : Y \to X$ (of locally Noetherian schemes), the pullback functor f^* (Lemma 2.10.19) preserves coherent sheaves (since if M is a finitely generated A-module, and $A \to B$ a ring homomorphism then $M \otimes_A B$ is a finitely generated B-module).

It is a consequence of the above properties of coherent modules that $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ is an abelian subcategory.

3.5.2 Cohomology of coherent sheaves on projective schemes

Theorem 3.5.4. Let $i : Y \subset X := \mathbf{P}^n_A$ be a closed subscheme, with A being a Noetherian ring. For any $F \in \mathrm{Coh}(Y)$, the cohomology groups $\check{\mathrm{H}}^*(Y, F)$ are finitely generated A-modules. In particular, $F(Y) = \check{\mathrm{H}}^0(Y, F)$ is finitely generated.

By Lemma 3.4.4, and using that $i_*F \in \mathrm{Coh}(\mathbf{P}^n_A)$ (Example 3.5.3), it is enough to prove Theorem 3.5.4 in the case $Y = X = \mathbf{P}^n_A$.

Lemma 3.5.5. For any $F \in \mathrm{Coh}(X)$ there is a surjection
$$\bigoplus_{j=1}^{n} \mathcal{O}_X(-d_j) \to F$$
for appropriate (finite) n and $d_j \in \mathbf{Z}$.

Proof. We need to find a map of coherent sheaves whose cokernel in $\mathrm{QCoh}(X)$ or $\mathrm{Coh}(X)$ is 0; equivalently, we need to supply a map such that the sections on every basic open $U_i := D_+(t_i)$, $\bigoplus_j \mathcal{O}_X(U_i) \to F(U_i)$ are surjective. Since this involves finitely many i, and since $F(U_i) =: M_i$ is a finitely generated $\mathcal{O}_X(U_i)$-module, it is enough to prove the following: for any i and any fixed element $s \in F(U_i)$, there is some d such that $t_i^d s \in (F(d))(U_i)$ extends to a global section s' of $F(d)$, i.e., an element in $F(d)(X)$. Here we use Lemma 2.10.3, i.e., a global section $s' \in (F(d))(X)$ is nothing but a map (of $\mathcal{O}_X$-modules) $\mathcal{O}_X \to F(d)$ or, equivalently, a map $\mathcal{O}_X(-d) \to F$. This assertion is precisely the content of Exercise 2.10.28. □

Proof. (of Theorem 3.5.4, for $Y = X = \mathbf{P}^n_A$) We will prove that $\check{\mathrm{H}}^q(X, F)$ is finitely generated by descending induction on q. The statement holds for $q > n + 1$ by Exercise 3.3.7.

Pick a surjection as above, and consider its kernel
$$0 \to K \to \bigoplus_j \mathcal{O}_X(-d_j) \to F \to 0. \tag{3.5.6}$$

The kernel K is coherent since $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ is an abelian subcategory (this uses that A is Noetherian). Consider the long exact cohomology sequence

$$\check{\mathrm{H}}^q(X, \bigoplus \mathcal{O}(-d_j)) = \bigoplus_j \mathrm{H}^q(X, \mathcal{O}(-d_j)) \to \check{\mathrm{H}}^q(X, F) \to \check{\mathrm{H}}^{q+1}(X, K) \to \dots.$$

The outer terms are finitely generated by Theorem 3.4.1 and the inductive assumption, respectively. Hence the middle group is also finitely generated (over the *Noetherian* ring A). □

Remark 3.5.7. The proof technique above also shows that for a fixed $F \in \mathrm{Coh}(\mathbf{P}^n_A)$, there is $e \gg 0$ such that for all $e' \geqslant e$ we have

$$\check{\mathrm{H}}^q(X, F(e')) = 0 \text{ for all } q > 0.$$

The following statement is the algebro-geometric incarnation of *Liouville's theorem* (which asserts that a holomorphic function on a connected compact complex manifold is constant).

Corollary 3.5.8. Let $X \subset \mathbf{P}^n_k$ be an integral closed subscheme, where k is algebraically closed. (In more classical terminology, one refers to this by saying that X is a *projective variety*, where "variety" means integral and of finite type scheme over $k = \overline{k}$.) Then

$$\mathrm{H}^0(X, \mathcal{O}_X) = k. \tag{3.5.9}$$

Proof. By Theorem 3.5.4, $A := \mathrm{H}^0(X, \mathcal{O}_X)$ is a finite-dimensional k-vector space. By Lemma 2.2.2 it is a domain. Thus A is a finite field extension of k, and therefore $A = k$, since $k = \overline{k}$. □

Remark 3.5.10. Note that both the closed embedding $X \subset \mathbf{P}^n_k$ and the structural map $\mathbf{P}^n_k \to \operatorname{Spec} k$ are proper, and hence so is the map $X \to \operatorname{Spec} k$ (Lemma 2.9.9). The assertion of Theorem 3.5.4 holds true for arbitrary proper morphisms $X \to \operatorname{Spec} B$. This is proved in [Gro61, Théorème 3.2.1] using Chow's lemma (stated above in Proposition 2.9.17). Also note that Corollary 3.5.8 reproves the statement in Exercise 2.9.22 (in the projective case). Finally, the assumption that k is algebraically closed can be relaxed, as discussed in Exercise 3.5.12.

Exercises

Exercise 3.5.11. We say that some $F \in \mathrm{QCoh}(X)$ is *generated by global sections* iff there is a surjection (in the category $\mathrm{QCoh}(X)$, see around (3.2.8) what this means)

$$\bigoplus_{i \in I} \mathcal{O}_X \to F.$$

Let now $X = \mathbf{P}^n_A$ with A Noetherian and $F \in \mathrm{Coh}(X)$. Prove that there is an $e \gg 0$ such that for all $e' \geqslant e$ the sheaf $F(e')$ is generated by finitely many global sections (i.e., the sum above is finite).

One refers to this statement by saying that $\mathcal{O}_X(1)$ is an *ample line bundle*. This is an important positivity property in algebraic geometry.

Exercise 3.5.12. A scheme X/k is called *geometrically integral* if $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ (where $\overline{k}$ is an algebraic closure of k) is integral.

(1) Prove that any geometrically integral scheme is integral.

(2) The converse does not hold: for $k = \mathbf{R}$, prove that $\operatorname{Spec} \mathbf{R}[t]/f$ is geometrically integral iff $\deg f = 1$.

(3) Prove that the natural map $k \to \mathrm{H}^0(X, \mathcal{O}_X)$ (which is nothing but the pullback along the structural map $X \to \operatorname{Spec} k$) is an isomorphism provided that X is geometrically integral, and $X \subset \mathbf{P}^n_k$ is a closed subscheme.

Exercise 3.5.13. (Solution at p. 109) For the purposes of this exercise, we call a function

$$P : \mathbf{N} \to \mathbf{N}$$

a *numerical polynomial* if there is a (necessarily unique) polynomial $Q \in \mathbf{Q}[x]$ and $m_0 \in \mathbf{N}$ such that

$$P(m) = Q(m)$$

for all $m \in \mathbf{N}$, $m \geqslant m_0$. We set its *degree* to be $\deg P := \deg Q$.

Let k be a field and let us write $X := \mathbf{P}^n_k$. For $F \in \mathrm{Coh}(X)$, we consider the function

$$P := P_F : \mathbf{N} \to \mathbf{N}, P(m) := \dim_k \mathrm{H}^0(X, F(m)).$$

(1) For $e \in \mathbf{Z}$, compute the values of the function $P_{\mathcal{O}_X(e)}$. You may use without proof that the k-vector space of homogeneous polynomials of degree r in s variables has dimension $\binom{r+s-1}{s-1}$. Compute $\deg P_{\mathcal{O}_X}$.

(2) Let $i : Y = V(f) \subset X$ be the closed immersion defined by a non-zero homogeneous polynomial $f \in k[t_0, \dots, t_n]_d$ of degree $d \geqslant 0$. Prove that

$$\deg P_{i_* \mathcal{O}_Y} \leqslant n - 1$$

(you may use (3) below). Is it possible that "<" holds in the above?

(3) For any $F \in \mathrm{Coh}(X)$, prove that there is some $e_0 \geqslant 0$ such that

$$P_F(e) = \chi(X, F(e)) \tag{3.5.14}$$

for $e \geqslant e_0$.

(4) (Bonus) Generalizing Part (2), prove that for $Y = V(f_1, \dots, f_m)$, for homogeneous polynomials $f_1, \dots, f_m \in k[t_0, \dots, t_n]$, we have

$$\deg P_{i_* \mathcal{O}_Y} \leqslant n - m.$$

3.6 Outlook: The Riemann–Roch theorem

In this outlook, we are going to state the Riemann–Roch theorem, which is the cornerstone in the theory of algebraic curves. Throughout we make the following assumptions:

Notation 3.6.1.
- $k = \overline{k}$ will be an algebraically closed field,
- X is a scheme that is
  1. 1-dimensional (Definition 1.3.1),
  2. integral, i.e., reduced and irreducible (Definition 2.2.1),
  3. a closed subscheme of some $\mathbf{P}^n_k$,
  4. *regular* in the sense that the local rings $\mathcal{O}_{X,x}$ are regular rings. Note: if x is the generic point, this condition is vacuous. If x is a closed point, so that $\dim \mathcal{O}_{X,x} = 1$, this means that $\mathfrak{m}_x$ is a principal ideal, so that equivalently for these points, $\mathcal{O}_{X,x}$is a discrete valuation ring, see around Corollary 1.7.17). We the *valuation* map

     $$\mathrm{val} := \mathrm{val}_x : Q(\mathcal{O}_{X,x}) \backslash \{0\} \to \mathbf{Z}$$

     which is the unique group homomorphism sending a generator $\varpi_x \in \mathfrak{m}_x$ to 1, and sending all the elements in $\mathcal{O}^\times_{X,x}$ to 0. (I.e., there is a unique $u \in \mathcal{O}^\times_{X,x}$ and $n \in \mathbf{Z}$ such that $s = \varpi^n_x \cdot u$, and then $\mathrm{val}(s) = n$).

We refer to the conjunction of these assumptions by saying that X is a *smooth projective curve*. We will denote by

$$K := k(X) := \mathcal{O}_{X,\eta}$$

the local ring at the generic point (for any nonempty open $U \subset X$, we have $K = Q(\mathcal{O}_X(U))$). We refer to its elements as *rational functions*.

According to Corollary 3.5.8, we have $\dim_k \mathcal{O}_X(X) = 1$, paralleling the fact that on a connected compact Riemann surface there are only constant holomorphic functions. However, there is an infinite-dimensional space of meromorphic functions (i.e., locally of the form $z^{-n} \cdot f(z)$, where $f(z)$ is holomorphic). The Riemann–Roch theorem below gives a way to count, more precisely, how many functions having prescribed pole orders there are. In this section, we formulate this theorem, including the rudiments of the necessary preliminaries in the specific situation of a smooth projective curve.

Definition 3.6.2. For X as above, the *genus* is defined as

$$g := g(X) := \dim_k \mathrm{H}^1(X, \mathcal{O}_X).$$

We will use without proof that for any $F \in \mathrm{Coh}(X)$:

$$\mathrm{H}^n(X, F) = 0 \text{ for } n > 1 = \dim X. \tag{3.6.3}$$

(This can be proved by in two ways: 1) by comparing Cech cohomology with sheaf cohomology and using a vanishing for sheaf cohomology beyond the dimension [Stacks, Tags 03AG, 04AR], or 2) in our situation by Serre duality as stated in Theorem 3.6.30 below.) Thus

$$1 - g(X) = \dim \mathrm{H}^0(X, \mathcal{O}_X) - \dim \mathrm{H}^1(X, \mathcal{O}_X) = \chi(X, \mathcal{O}_X).$$

Example 3.6.4. If $i : X = V(f) \subset \mathbf{P}^2$ is a plane curve, where f is a homogeneous polynomial of degree d, then we have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-d) \xrightarrow{f} \mathcal{O}_{\mathbf{P}^2} \to i_*\mathcal{O}_X \to 0,$$

and

$$\chi(\mathbf{P}^2, i_*\mathcal{O}_X) = \chi(X, \mathcal{O}_X) = \chi(\mathbf{P}^2, \mathcal{O}) - \chi(\mathbf{P}^2, \mathcal{O}(-d)) = 1 - \binom{2-d}{2}.$$

For $d \geqslant 0$, we have $\chi(\mathbf{P}^2, \mathcal{O}(-d)) = \dim_k \mathrm{H}^2(\mathbf{P}^2, \mathcal{O}(-d))$, and by (3.4.2) we see that $\mathrm{H}^n(\mathbf{P}^2, \mathcal{O}(-d))$ is isomorphic (as a k-vector space) to the space of homogeneous polynomials (in 3 variables) of degree $d-3$, and the dimension of this k-vector space equals $\binom{(d-3)+(3-1)}{3-1} = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$. Hence

$$g(X) = \chi(X, \mathcal{O}_X) - 1 = \chi(\mathbf{P}^2, \mathcal{O}(-d)) = \frac{(d-1)(d-2)}{2}.$$

For example, if $\deg f = d = 3$, we obtain $g(X) = 1$, while for $d = 1$ or $d = 2$, we have $g = 0$. For X as in Notation 3.6.1 and $k = \mathbf{C}$, one can prove that there are equalities

$$2 \dim \mathrm{H}^1(X, \mathcal{O}_X) = 2 \dim \mathrm{H}^0(X, \Omega^1) = \dim \mathrm{H}^1_{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Q}),$$

where at the right we have the first singular cohomology group of the complex submanifold of $\mathbf{P}^2_{\mathbf{C}}$ defined by the equation f. One may conclude that g is equal to the number of "handles" attached to $S^2 \cong \mathbf{P}^1_{\mathbf{C}}$.

Definition 3.6.5. (see [Vak17, §14.2], [Stacks, Tag 0BE2] for the definition in general) A *Weil divisor* D on X is a finite formal linear sum of closed points, written as $D := \sum n_i x_i$, where $x_i \in X$ is a closed point and $n_i \in \mathbf{Z}$. In other words, $D \in \bigoplus_{x \in X^{\mathrm{cl}}} \mathbf{Z}$. The *degree* of such a Weil divisor is

$$\deg D := \sum n_i \in \mathbf{Z}.$$

The *support* of the divisor is

$$|D| := \bigcup_{n_x \neq 0} \{x\}.$$

Definition and Lemma 3.6.6. For a Weil divisor $D = \sum n_x x$, we define

$$\mathcal{O}(D) := \mathcal{O}_X(D)$$

to be the sheaf

$$\mathcal{O}(D)(U) := \{s \in k(X)^\times | \mathrm{val}_x(s) + n_x \geqslant 0 \text{ for all } x \in U\} \cup \{0\}.$$

(In other words, we consider rational functions that have a pole of order at most n_x in the points x.) For another divisor $E = \sum m_x x$ and their sum $D+E := \sum (n_x + m_x)x$, we have an isomorphism

$$\mathcal{O}(D+E) = \mathcal{O}(D) \otimes \mathcal{O}(E).$$

Thus $\mathcal{O}(D)$ is a line bundle (and therefore in particular a coherent sheaf) with dual given by

$$\mathcal{O}(D)^\vee = \mathcal{O}(-D).$$

Proof. On $X \backslash |D|$, we have $\mathcal{O}(D) \cong \mathcal{O}$. For any of the points $x \in |D|$ there is an open neighborhood $U \ni x$ such that $U \cap |D| = \{x\}$, such that $\varpi_x \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ extends to an element of $f \in \mathcal{O}_X(U)$, and such $f|_{U \backslash \{x\}}$ is invertible. Then the multiplication map $\mathcal{O}_X|_U \stackrel{f^{-n_x}}{\to} \mathcal{O}(D)|_U$ is an isomorphism. □

Lemma 3.6.7. (See [Har83, §II.6], especially Corollary 6.16 there, or [Vak17, Proposition 14.2.10] for the statement in the generality of a Noetherian, integral, separated scheme whose local rings $\mathcal{O}_{X,x}$ are factorial rings) The map

$$\bigoplus_{x \in X^{\mathrm{cl}}} \mathbf{Z} \to \mathrm{Pic}(X), D \mapsto \mathcal{O}(D)$$

induces an isomorphism of abelian groups

$$\mathrm{Cl}(X) \stackrel{\cong}{\to} \mathrm{Pic}(X),$$

where

$$\mathrm{Cl}(X) := \bigoplus_{x \in X^{\mathrm{cl}}} \mathbf{Z} / \left(\mathrm{div}(s), s \in k(X)^\times\right)$$

is the so-called *divisor class group*, where

$$\mathrm{div}(s) := \sum_{x \in X^{\mathrm{cl}}} \mathrm{val}_x(s) x.$$

Moreover, we have

$$\deg(\mathrm{div}(s)) = 0,$$

so there is a well-defined map

$$\deg : \mathrm{Cl}(X) \to \mathbf{Z}.$$

Example 3.6.8. Recall from Theorem 2.11.4 that on $X = \mathbf{P}^1_k$, any line bundle is of the form $\mathcal{O}(e)$ (Definition 2.10.14), i.e.

$$\mathrm{Pic}(\mathbf{P}^1_k) = \mathbf{Z}.$$

This is matched by the fact that any two (closed) points $x, x' \in (\mathbf{P}^1_k)^{\mathrm{cl}}$ are *rationally equivalent*, i.e., there is a function $s \in k(\mathbf{P}^1_k) = k(t)$ such that $\mathrm{div} s = x - x'$. (This is seen by showing that there is an automorphism of $\varphi : \mathbf{P}^1 \to \mathbf{P}^1$ such that $\varphi(x) = 0, \varphi(x') = 1$ and then the function $\frac{t}{t-1}$ has a simple zero at 0 and a simple pole at 1; cf. [Vak17, Exercise 16.4.B]).

Definition 3.6.9. For a divisor D and a line bundle L on X we write

$$h(L) := \dim_k \mathrm{H}^0(X, L)$$

and $h(D) := h(\mathcal{O}(D))$.

Lemma 3.6.10. (See [Har83, Proposition II.7.7] for a discussion in the generality of a smooth projective variety.) If $h(D) \neq 0$, then $\mathcal{O}(D) \cong \mathcal{O}(D')$ with D' being an *effective divisor*, i.e., $D' = \sum n_i x_i$ with $n_i \geqslant 0$.

Proof. If we have a non-zero global section $s \in \mathcal{O}(D)(X)$, then locally on $U \subset X$ where $\mathcal{O}(D)|_U \cong \mathcal{O}_X|_U$, s gives rise to a section $s' \in \mathcal{O}_X(U)$. This section depends on the choice of the isomorphism up to multiplying with some element in $\mathcal{O}_X(U)^\times$, so that $V(s')$ is a well-defined closed subset of dimension 0 (since $s \neq 0$). This subset therefore defines an effective divisor D', and one checks $\mathcal{O}(D) \cong \mathcal{O}(D')$. □

Proposition 3.6.11. (Riemann–Roch theorem, preliminary version) Let X be as in Notation 3.6.1, $g := g(X)$ its genus and D a divisor on X. Then the Euler characteristic of $\mathcal{O}(D)$ can be computed by

$$\chi(X, \mathcal{O}(D)) = \deg D + 1 - g.$$

Proof. Using (3.6.3), we have to prove

$$\dim \mathrm{H}^0(X, \mathcal{O}(D)) - \dim \mathrm{H}^1(X, \mathcal{O}(D)) = \deg D + 1 - g. \tag{3.6.12}$$

For $D = 0$, this formula holds true by the definition of g and by (3.5.9). It suffices to show that (3.6.12) holds for some divisor D iff it holds for $D' = D + x$, where $x \in X^{\mathrm{cl}}$ is any (closed) point. We claim that there is an exact sequence (in $\mathrm{QCoh}(X)$)

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D + x) \to i_* k \to 0, \tag{3.6.13}$$

where $i : \{x\} \to X$ is the closed embedding, and we have $x = \operatorname{Spec} k$ (by Hilbert's Nullstellensatz), and we have written $k = \mathcal{O}_{\operatorname{Spec} k}$. Indeed, the restriction of the above sequence to $X \backslash \{x\}$ is exact, given that $(i_* k)|_{X\backslash\{x\}} = 0$. Consider now an open affine neighborhood $U = \operatorname{Spec} A \ni x$ as in the proof of Definition and Lemma 3.6.6, i.e., U contains no other point of D, and $\varpi_x \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ extends to an element of $\varpi \in A = \mathcal{O}_X(U)$ such that $x = V(\varpi)$. The restriction of the above sequence to U then arises by applying $\widetilde{\ }$ to the following exact sequence of A-modules (where in the middle $\frac{1}{\varpi} A \subset Q(A)$ denotes the A-submodule generated by $\frac{1}{\varpi}$):

$$0 \to A \subset \frac{1}{\varpi} \cdot A \to \frac{1}{\varpi} \cdot A/A \to 0$$

which is isomorphic to

$$0 \to A \xrightarrow{\varpi} A \to A/\varpi \to 0.$$

This confirms (3.6.13), so we obtain our claim from the additivity of Euler characteristics in (3.2.16), given that $\chi(X, i_* k) = \chi(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k}) = 1$. □

3.6.1 Interlude: Kähler differentials

Definition 3.6.14. For an A-algebra B, the *Kähler differentials* is the B-module

$$\Omega_{B/A} := \bigoplus_{b \in B} B db / \sim,$$

where db is just a symbol for each $b \in B$, and the relations divided out are

$$\begin{aligned} d(bb') &= b \cdot db' + b' \cdot db \quad (\textit{Leibniz rule}) \\ d(ab + a'b') &= a \cdot db + a' \cdot db'. \end{aligned}$$

for $a, a' \in A$, $b, b' \in B$.

Remark 3.6.15. One should think of $\Omega_{B/A}$ as being the algebraic analogue of differential 1-forms. The above relations imply

$$\begin{aligned} d1 &= d1 + d1 \Rightarrow d1 = 0 \\ da &= ad1 = 0. \end{aligned} \tag{3.6.16}$$

There is a natural map (called *universal derivation*, in view of Exercise 3.6.27 it corresponds to $\mathrm{id}_B \in \mathrm{Der}_A(B, B)$)

$$d : B \to \Omega_{B/A}, b \mapsto db. \tag{3.6.17}$$

Example 3.6.18. Let $B = A[t_i, i \in I]$, where I is some (possibly even infinite) index set, so $\operatorname{Spec} B = \mathbf{A}^I_A$. Then

$$\Omega_{B/A} = \bigoplus_{i \in I} B dt_i,$$

i.e., the Kähler differentials for a polynomial algebra are just a free module (over that polynomial algebra). (This computation matches the analogous fact in differential geometry, where the 1-forms on $\mathbf{R}^n$ are of the form

$$\omega = \sum_{i=1}^n f_i dx_i.)$$

The universal derivation is given by

$$d(f) = \sum_i \frac{\partial f}{\partial t_i} dt_i.$$

One quick way to see this is to first prove Exercise 3.6.27 and to show $\mathrm{Der}_A(A[t_i], M) = \prod_i M$.

Lemma 3.6.19. (*First fundamental sequence*) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be two ring homomorphisms. Then there is an exact sequence (of C-modules)

$$\Omega_{B/A} \otimes_B C \xrightarrow{v} \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

The maps are given by $v : db \otimes c \mapsto c \cdot d(g(b))$ and $dc \mapsto dc$, respectively.

Proof. The right hand map is clearly surjective (since symbols dc generate $\Omega_{C/B}$). The extra relations $db = 0$ in $\Omega_{C/B}$, for $b \in B$ are precisely the image of the generators $db \otimes 1$ in the left hand group. $\square$

Lemma 3.6.20. (*Conormal sequence* or *second fundamental exact sequence*) Suppose $A \to B \xrightarrow{\pi} C = B/I$ are two ring homomorphisms (with I an ideal in B). Then there is an exact sequence of C-modules

$$I/I^2 \xrightarrow{1 \otimes d} C \otimes_B \Omega_{B/A} \xrightarrow{D\pi} \Omega_{C/A} \to 0.$$

Here, the left hand map is $i \mapsto 1 \otimes d(i)$, where d is the universal derivation (3.6.17).

Proof. First note that d factors over I/I^2 since $d(ii') = 1 \otimes (idi' + i'di) = i \otimes di' + i' \otimes di = 0 \in C \otimes_B \Omega_{B/A} = \Omega_{B/A}/I\Omega_{B/A}$.

We already noted that $D\pi$ is surjective since $C \to B$ is surjective. To prove the exactness in the middle we use again Exercise 3.6.27 and Exercise 3.6.28 and prove that we have the following exact sequence, for any C-module T:

$$\begin{tikzcd} \mathrm{Hom}_C(\Omega_{C/B}, T) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, T) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_C(I/I^2, T) \arrow[d, equal] \\ \mathrm{Der}_B(B/I, T) \arrow[r] & \mathrm{Der}_A(B, T) \arrow[r, "\partial \mapsto \partial|_I"] & \mathrm{Hom}_B(I, T). \end{tikzcd}$$

For the right hand identification note that applying $\otimes_B I$ to $0 \to I \to B \to C \to 0$ gives the exact sequence $0 \to I^2 (= \mathrm{im}\, I \otimes_B I) \to I \to C \otimes_B I \to 0$. This sequence is clearly exact. $\square$

Example 3.6.21. Suppose $A = k$ is a field, $B = k[x, y, t]$, so $\operatorname{Spec} B = \mathbf{A}^3$ and $C = k[x, y]/f$. In other words $\operatorname{Spec} C = V(f)$ is the zero-set of f. By Theorem 1.3.4, we have $\dim C = 1$.

The conormal sequence implies the following formula

$$\Omega_{C/k} = (Bdx \oplus Bdy)/(f, df).$$

We specialize to the case

$$f(x, y) = y^2 - (x^3 + ax + b),$$

where $a, b \in k$ are fixed. The above quotient is modding out $2ydy$ and $(3x^2 + a)dx$. We are interested in seeing when the stalk of the above C-module at a closed point $(x - r, y - s) \in \operatorname{Spec} C$ (with $r, s \in k$), i.e., $s^2 = r^3 + ar + b$, is 1. If $s \neq 0$ (and $\operatorname{char} k \neq 2$) then $2ydy \neq 0$ (and it follows that $3x^2 + a \neq 0$). If $s = 0$, then we have $0 = r^3 + ar + b$, and the above module is of rank 1 precisely if $3r^2 + a \neq 0$, i.e., if r is a *simple* zero of the polynomial $x^3 + ax + b$. It is known from the theory of (cubic) polynomials that this happens precisely if the so-called *discriminant* $\Delta := 4a^3 + 27b^2 \neq 0$.

Proposition 3.6.22. (*Jacobian criterion* for smoothness, see e.g. [Eis95, Theorem 16.19] for the statement for $k[t_i]/(f_j)$) Fix $C = k[x, y]/f$, and $\mathfrak{p} \in \operatorname{Spec} C$. We assume that $k(\mathfrak{p})$ is a separable extension of k (this is automatic if $\operatorname{char} k = 0$ or if $k = \overline{k}$).

Then $C_{\mathfrak{p}}$ is a DVR iff the vector

$$(\partial f/\partial x, \partial f/\partial y)$$

is non-zero in $k(\mathfrak{p})^2$. In this event, the above conormal exact sequence is split exact, i.e., there is a split exact sequence of C-modules

$$0 \to (f)/(f^2) \to C \otimes_B \Omega_{B/k} \to \Omega_{C/k} \to 0.$$

In order to establish Kähler differentials as a quasi-coherent sheaf, we need the following compatibility with localizations.

Lemma 3.6.23. Let $A \to B$ be a ring homomorphism and $S \subset B$ a multiplicatively closed subset, the natural map

$$B[S^{-1}] \otimes_B \Omega_{B/A} \to \Omega_{B[S^{-1}]/A}$$

is an isomorphism. Under this isomorphism $-s^{-2}ds$ at the left corresponds to $d(1/s)$ at the right, for $s \in S$.

Proof. It suffices to show that $\operatorname{Hom}_{B[S^{-1}]}(-, M)$ gives an isomorphism for each $B[S^{-1}]$-module M. This corresponds to restriction of derivations:

$$\operatorname{Der}_A(B[S^{-1}], M) \to \operatorname{Der}_A(B, M).$$

Given an A-linear derivation $\partial : B \to M$ one checks that the map $\partial' : B[S^{-1}] \to M$ defined by

$$\partial'(\frac{b}{s}) := \frac{1}{s}\partial b - b\frac{1}{s^2}\partial s$$

is well-defined, is a derivation, and is the unique derivation extending ∂. □

Definition and Lemma 3.6.24. Let $X \to Y = \operatorname{Spec} A$ be a map of schemes. Then there is a unique quasi-coherent sheaf $\Omega_{X/Y}$, called the sheaf of *Kähler differentials* whose restriction to an open affine subscheme $U = \operatorname{Spec} B \subset X$ satisfies

$$\Omega_{X/Y}|_U = \widetilde{\Omega_{B/A}}.$$

Corollary 3.6.25. If $i : X \subset Y$ is a closed embedding of k-schemes, and $I = \ker(\mathcal{O}_Y \to i_* \mathcal{O}_X) \in \mathrm{QCoh}(Y)$ is the so-called *ideal sheaf* defining i, there is an exact sequence

$$I/I^2 \to i^* \Omega_{Y/k} \to \Omega_{X/k} \to 0.$$

If X satisfies the condition ((4)) in Notation 3.6.1 and $Y = \mathbf{P}^n_k$, then this sequence can be extended to an exact sequence

$$0 \to I/I^2 \to i^* \Omega_{Y/k} \to \Omega_{X/k} \to 0. \tag{3.6.26}$$

In this event $\Omega_{X/k}$ is a locally free sheaf of rank $1 = \dim X$.

Exercises

Exercise 3.6.27. For any B-module M, establish a natural bijection

$$\mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M)$$

where at the right we have the set (or, rather A-module) of *derivations*, defined by

$$\mathrm{Der}_A(B, M) = \{f : B \to M (\text{map of } A\text{-modules}), f(bb') = bf(b') + b'f(b) \text{ for all } b, b' \in B\}.$$

Exercise 3.6.28.

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

be two composable morphisms of modules (over some fixed ring A).

(1) Prove that the sequence is exact if the induced sequence

$$\mathrm{Hom}(M'', N) \xrightarrow{-\circ g} \mathrm{Hom}(M, N) \xrightarrow{-\circ f} \mathrm{Hom}(M', N)$$

is exact for any A-module N. (Hint: it is enough to take $N = M/\operatorname{im} f$.)

(2) Prove that the sequence is *split exact* (i.e., there is an isomorphism $M \cong M' \oplus M''$ such that f becomes the canonical injection $M' \to M' \oplus M''$ and g the canonical projection) if

$$0 \to \mathrm{Hom}(M'', N) \xrightarrow{-\circ g} \mathrm{Hom}(M, N) \xrightarrow{-\circ f} \mathrm{Hom}(M', N) \to 0$$

is exact.

3.6.2 The statement of Riemann–Roch

We continue working under the assumptions in Notation 3.6.1. By these assumptions and by the Jacobian criterion for smoothness (cf. Proposition 3.6.22 in the case of a plane curve), the sheaf $\Omega_{X/k}$ is *locally free of rank 1*, i.e., a line bundle.

By Lemma 3.6.7, we can therefore find a divisor K (which is well-defined only up to *rational equivalence*, i.e., up to replacing K by $K + \mathrm{div}(s)$, where $s \in k(X)^\times$) such that

$$\mathcal{O}(K) = \Omega_{X/k}.$$

This divisor is called *canonical divisor*.

Example 3.6.29. If $i : X = V(f) \subset \mathbf{P}^2_k = \operatorname{Proj} k[t_0, t_1, t_2]$ is a plane curve with $\deg f = d$ (satisfying our standing assumptions in Notation 3.6.1), then

$$K = (d-3)(X \cap V(t_2)),$$

say, where $\cap$ denotes the scheme-theoretic intersection (Exercise 2.6.5). This can be deduced from the exactness of the conormal sequence in (3.6.26), which by passing to the highest exterior powers gives

$$\Omega_{X/k} \otimes (I/I^2) = i^* \Omega_{\mathbf{P}^2_k/k}.$$

One computes the right hand side to be $\mathcal{O}_{\mathbf{P}^2}(-3)$, and $(f)/(f^2) = i^* \mathcal{O}(-d)$.

Then 3.6.12 allows to compute $\dim \mathrm{H}^0(X, \mathcal{O}_X(D))$ in terms of computable data ($\deg D$, $g = g(X)$) and an "error term" $\mathrm{H}^1(X, \mathcal{O}_X(D))$. This error term can be accessed as follows.

Theorem 3.6.30. (*Serre duality*) For X as in Notation 3.6.1 and any coherent sheaf $F \in \mathrm{Coh}(X)$ there is an isomorphism

$$\mathrm{H}^i(X, F) \cong \left(\mathrm{H}^{1-i}(X, \underline{\mathrm{Hom}}(F, \Omega_{X/k})\right)^\vee$$

(where at the right $\vee$ denotes the dual k-vector space). In particular, for $F = \mathcal{O}(D)$, we have

$$\mathrm{H}^1(X, \mathcal{O}_X(D)) = \left(\mathrm{H}^0(X, \mathcal{O}(K - D)\right)^\vee.$$

Remark 3.6.31. This holds true much more generally: for $X \subset \mathbf{P}^n_k$ (closed) being smooth over k and integral of dimension d, we have an isomorphism

$$\mathrm{H}^i(X, F) \cong \left(\mathrm{H}^{d-i}(X, \underline{\mathrm{Hom}}(F, \bigwedge^d \Omega_{X/k}))\right)^\vee.$$

See, e.g., [Vak17, Theorem 18.5.1] or [Stacks, Tag 0DWE], especially [Stacks, Tag 0BRT] for an even more sweeping account.

Example 3.6.32. For $X = \mathbf{P}^1 = \mathrm{Proj}\, k[t_0, t_1] =: \mathrm{Proj}\, A$, Serre duality for the sheaves $F = \mathcal{O}_X(e)$ is nothing but the agreement

$$\mathrm{H}^0(X, \mathcal{O}(e)) = A_e$$

with the dual of $\mathrm{H}^1(X, \mathcal{O}(-2 - e))$, cf. (3.4.2).

Theorem 3.6.33. (*Riemann–Roch*, final form) For X as in Notation 3.6.1, and any divisor D on X we have an equality

$$\dim \mathrm{H}^0(X, \mathcal{O}(D)) - \dim \mathrm{H}^0(X, \mathcal{O}(K - D)) = \deg D + 1 - g.$$

Example 3.6.34. Putting $D = K$ and using Serre duality we get

$$\begin{aligned} \deg K &= g - 1 + \chi(X, \mathcal{O}(K)) \\ &= g - 1 - \chi(X, \mathcal{O}) \\ &= g - 1 - (1 - g) \\ &= 2g - 2. \end{aligned}$$

Corollary 3.6.35. If D is a divisor with $\deg D > \deg K = 2g - 2$, then

$$\dim \mathrm{H}^0(X, \mathcal{O}(D)) = \deg D + 1 - g.$$

Proof. This holds since $\mathrm{H}^1(X, \mathcal{O}(D)) = (\mathrm{H}^0(X, \mathcal{O}(K - D)))^\vee = 0$ by Lemma 3.6.10 given that $\deg K - D < 0$. $\square$

Chapter 4

Solutions for selected exercises

Solution of Exercise 1.1.21: We first prove that $Y \subset X$ (in any topological space X) is irreducible iff its closure $\overline{Y}$ is irreducible. First, we have $Y \neq \varnothing$ iff $\overline{Y} \neq \varnothing$. Now, directly from the definition, a non-empty space X is irreducible iff any two non-empty open subsets $U_1, U_2 \subset X$ have non-empty intersection, i.e., $U_1 \cap U_2 \neq \varnothing$. Also note that the open subsets of Y (or $\overline{Y}$) are of the form $U \cap Y$ (resp. $U \cap \overline{Y}$), with $U \subset \operatorname{Spec} A$ open. Now, we observe that $U_1 \cap U_2 \cap Y \neq \varnothing$ iff $U_1 \cap U_2 \cap \overline{Y} \neq \varnothing$, by definition of the closure. Given that $\overline{Y} = V(I(Y))$, this claim allows us to replace Y by its closure, so we may assume Y is closed.

To prove the Parts (1) and (2) of the exercise, we use the bijection established in Lemma 1.1.6, under which the non-empty closed subsets $\varnothing \neq Y \subset \operatorname{Spec} A$ correspond to proper radical ideals $I \subsetneq A$. We may replace A by $A/I(Y)$, so we are reduced to proving that for a ring A, $\operatorname{Spec} A$ is irreducible iff $\sqrt{0} = I(\operatorname{Spec} A)$ is a prime ideal.

"$\Leftarrow$": given two open nonempty subsets $U_1, U_2 \subset \operatorname{Spec} A$, we need to prove $U_1 \cap U_2 \neq \varnothing$. By definition of the Zariski topology there are $f_1, f_2 \in A$ with $\varnothing \neq D(f_i) \subset U_i$. We then claim $D(f_1) \cap D(f_2) = D(f_1 f_2) \neq \varnothing$. Otherwise we would have $f_1 f_2 \in \bigcap_{\mathfrak{p} \subset A} \mathfrak{p} = \sqrt{0}$ (the nilradical), which is by assumption a prime ideal, so that $f_1 \in \sqrt{0}$, say. Then $D(f_1) = \varnothing$, which is a contradiction.

"$\Rightarrow$": if $\operatorname{Spec} A$ is irreducible, we show $\sqrt{0}$ is a prime ideal: if $a, b \in A$ are such that $(ab)^n = 0$ for $n \gg 0$, then $\operatorname{Spec} A \subset V((ab)^n) = V(ab) = V(a) \cup V(b)$, so by irreducibility $\operatorname{Spec} A = V(a)$, say, so that $a \in \sqrt{0}$.

Concerning Part (3), we note that $x^2 - y^2 = (x-y)(x+y)$ yields $V(x^2 - y^2) = V(x+y) \cup V(x-y)$ and these are two proper subsets, so $\operatorname{Spec} \mathbf{Z}[x,y]/x^2 - y^2$ is reducible (with two irreducible components being each isomorphic to $\operatorname{Spec} \mathbf{Z}[x] = \mathbf{A}^1$). By contrast, the polynomials $x^2 - y^3$ and $xy - 1$ are both irreducible (as one sees directly by considering the degree with respect to x), and therefore these polynomials cut out two irreducible (closed) subsets of $\mathbf{A}^2$.

Solution of Exercise 1.1.25: (solution provided by Cecilia Moriggi) Recall from (1.1.5) that $\bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \sqrt{0}$. Now fix $f \in A$: we have $\bigcap_{\mathfrak{p} \in \operatorname{Spec} A[f^{-1}]} \mathfrak{p} = \sqrt{0}_{A[f^{-1}]}$; recall that $\operatorname{Spec} A[f^{-1}]$ is in bijection with $D(f)$ and

$$\sqrt{0}_{A[f^{-1}]} = \left\{ \frac{a}{f^k} \text{s.t.} a \in A, \exists n \in \mathbf{N} : \left(\frac{a}{f^k}\right)^n = 0 \right\} = \{a \in A | \exists m \in \mathbf{N} : (af)^m = 0\} = \sqrt{0} : (f).$$

Thus we get $\bigcap_{\mathfrak{p} \in D(f)} \mathfrak{p} = \sqrt{0} : (f)$.

Now $\overline{D(f)} = V(I(D(f))) = \{\mathfrak{p} \in \operatorname{Spec} A | \bigcap_{\mathfrak{q} \in D(f)} \mathfrak{q} \subseteq \mathfrak{p}\}$ and so

$$\partial D(f) = \{\mathfrak{p} \in \operatorname{Spec} A | \{f\} \cup \bigcap_{\mathfrak{q} \in D(f)} \mathfrak{q} \subseteq \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} A | I \subseteq \mathfrak{p}\}$$

And from this we immediately get the required bijection.

4.1 Rings and their spectra

Solution of Exercise 1.3.7: (solution provided by Francesco Feltrin and Manuel Zorzo; this is also adressing Exercise 1.2.10) The inclusion $\mathbf{Z} \subset \mathbf{Z}[t]$ induces the map $\phi : \operatorname{Spec} \mathbf{Z}[t] \to \operatorname{Spec} \mathbf{Z}$, sending a prime ideal $\mathfrak{p}$ of $\mathbf{Z}[t]$ to $\mathfrak{p} \cap \mathbf{Z}$. The spectrum of $\mathbf{Z}[t]$ is the disjoint union of the fibers of ϕ: we will describe $\phi^{-1}(p\mathbf{Z})$ for $p\mathbf{Z} \in \operatorname{Spec} \mathbf{Z}$. We consider two cases, using in both cases Lemma 1.2.6 and the fact that prime ideals in $k[t]$ (for a field k such as $\mathbf{F}_p$ or $\mathbf{Q}$) are (0) and (f) with $f \in k[t]$ being an irreducible polynomial.

- $p \neq 0$ is a prime number. Then

$$\phi^{-1}(p\mathbf{Z}) \cong \operatorname{Spec}(\mathbf{Z}[t] \otimes_{\mathbf{Z}} k(p\mathbf{Z})) \cong \operatorname{Spec}(\mathbf{Z}[t] \otimes_{\mathbf{Z}} \mathbf{F}_p) \cong \operatorname{Spec} \mathbf{F}_p[t]$$

  We conclude that

$$\phi^{-1}(p\mathbf{Z}) = \{\mathfrak{p} = (p, f) | f \text{ is irreducible mod } p\} \cup \{p\mathbf{Z}[t]\},$$

- $p = 0$. Then

$$\phi^{-1}(0) \cong \operatorname{Spec}(\mathbf{Z}[t] \otimes_{\mathbf{Z}} k(0)) \cong \operatorname{Spec}(\mathbf{Z}[t] \otimes_{\mathbf{Z}} \mathbf{Q}) \cong \operatorname{Spec} \mathbf{Q}[t]$$

  We also note that an irreducible polynomial $f \in \mathbf{Q}[t]$ is, by clearing the denominators generating the same ideal (in $\mathbf{Q}[t]$) as a unique irreducible integer multiple of f having the property that its coefficients are coprime. We conclude that

$$\phi^{-1}(0) = \{\mathfrak{p} = (f) | f \in \mathbf{Z}[t] \text{ is an irreducible polynomial with coprime coefficients}\} \cup \{(0)\}.$$

Since $\mathbf{Z}[t]$ is a domain, (0) is a prime ideal, and it is the generic point (e.g., by using Exercise 1.1.21).

The ideals of type 4 are maximal (equivalently, the quotients $\mathbf{Z}[t]/(p, f)$ are fields); now it is easy to see that they are the only ones, because ideals of type 2 and 3 can always be included in one ideal of type 4 (alternatively, look at $\mathbf{Z}[t]/(p)$ and $\mathbf{Z}[t]/(f)$).

This discussion implies that $\dim \mathbf{Z}[t] = 2$: indeed we do have the chains of length 2: $(0) \subset (p) \subset (p, f)$ and $(0) \subset (f) \subset (p, f)$. And there are no longer chains, since there can't be inclusions between two ideals of type 2, or 3, or 4, nor between an ideal of type 2 and one of type 3, the only possible chains of length 2 are the ones just mentioned.

Solution of Exercise 1.3.8: An example is $A = V \times k^3$, where V is a DVR and k a field. For any ring A with 4 maximal ideals and one non-maximal (i.e., minimal), we have $\dim A = 1$.

Solution of Exercise 1.4.17: (Solution by Mario Mascolo and Paola Schiavone)

- "(2) $\Rightarrow$ (3)": Let $a \in A$ and let $\varphi \colon \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}] \to A$ be a ring homomorphism such that $\varphi(0, t) = a$. Set $e := \varphi(0, 1)$ and $u := \varphi(1, t)$. Note that, since $(0, 1)$ is an idempotent, so is e; and since $(1, t)$ is a unit, so is u. Finally,

$$a = \varphi(0, t) = \varphi(0, 1)\varphi(1, t) = eu.$$

- "(3) ⇒ (2)". Any element of $\mathbf{Z} \times \mathbf{Z}[t^{\pm 1}]$ can be written in the form $(n, p(t)t^{-\alpha})$, where $n \in \mathbf{Z}$, $\alpha \in \mathbf{N}$ and $p(t) \in \mathbf{Z}[t]$. Fix an homomorphism $\mathbf{Z}[t] \to A$ sending t to $a \in A$. By assumption, $a = ue$ for an idempotent $e \in A$ and a unit $u \in A^\times$. We define $\psi\colon \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}] \to A$ by setting

$$\psi(n, p(t)t^{-\alpha}) \coloneqq n(1-e) + p(a)u^{-\alpha}e, \tag{4.1.1}$$

for all $n \in \mathbf{Z}$, $\alpha \in \mathbf{N}$ and $p(t) \in \mathbf{Z}[t]$. It is clear that $\psi(0, t) = ae = uee = ue = a$. Some short computation proves that the map ψ defined in 4.1.1 is indeed a ring homomorphism. Indeed, ψ preserves sums (we assume without loss of generality, that $\beta \geqslant \alpha$):

$$\begin{aligned}\psi[(n, p(t)t^{-\alpha}) + (m, q(t)t^{-\beta})] &= \psi[n+m, (p(t)t^{\beta-\alpha} + q(t))t^{-\beta}] \\ &= (n+m)(1-e) + (p(a)a^{\beta-\alpha} + q(a))u^{-\beta}e \\ &= (n(1-e) + p(a)u^{-\alpha}e) + (m(1-e) + q(a)u^{-\beta}e) \\ &= \psi(n, p(t)t^{-\alpha}) + \psi(m, q(t)t^{-\beta});\end{aligned}$$

and it also preserves products:

$$\begin{aligned}\psi[(n, p(t)t^{-\alpha}) \cdot (m, q(t)t^{-\beta})] &= \psi(nm, p(t)q(t)t^{-\alpha-\beta}) \\ &= nm(1-e) + p(a)q(a)u^{-\alpha-\beta}e \\ &= (n(1-e) + p(a)u^{-\alpha}e) \cdot (m(1-e) + q(a)u^{-\beta}e) \\ &= \psi(n, p(t)t^{-\alpha})\psi(m, q(t)t^{-\beta}),\end{aligned}$$

where we have used the fact that e and $1-e$ are orthogonal idempotents. We conclude that ψ makes the diagram in (2) commute.

- Let R be any (commutative) ring. We say that two elements $x, y \in R$ are *associate* if they generate the same ideal, i.e. if there exists $u \in R^\times$ such that $y = xu$. Notice that if two idempotent elements of a ring are associate, they coincide. Indeed, let $e \in R$ and $f \in R$ be two idempotents such that $e = fu$ for some $u \in R^\times$. Then $f = u^{-1}e$, and

$$e = fu = f \cdot fu = feu^{-1}u = fe = u^{-1}e \cdot e = u^{-1}e = u^{-1}uf = f.$$

In particular, for any $r \in R$, either no idempotent is associate to r or the idempotent associate to r is *unique*.

- Using the previous observation we prove that $\gamma : \mathbf{Z}[t] \to \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}], t \mapsto (0, t)$ is an epimorphism (in the category of rings). Let us be given ring homomorphisms

$$\mathbf{Z}[t] \xrightarrow{\gamma} \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}] \overset{\varphi}{\underset{\psi}{\rightrightarrows}} A$$

be such that $\varphi \circ \gamma = \psi \circ \gamma$. To prove $\psi = \varphi$ it suffices to show that any ring homomorphism $\mathbf{Z} \times \mathbf{Z}[t^{\pm 1}] \to A$ is uniquely determined by the image of $(0, t)$.

Let $\varphi\colon \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}] \to A$ be a ring homomorphism. Set $a \coloneqq \varphi(0, t)$. We can write $a = eu$, where $e = \varphi(0, 1)$ is idempotent and $u = \varphi(1, t) \in A^\times$, hence a and e are associate in A: by what we have proven earlier, we may thus deduce that $\varphi(0, 1)$ is the *unique* idempotent element of A that is associate to $a = \varphi(0, t)$.

In particular, the value of $\varphi(0, 1)$ is uniquely determined by the value of $a = \varphi(0, t)$. Notice that $\varphi(1, 0) = \varphi(1, 1) - \varphi(0, 1) = 1 - \varphi(0, 1)$ and that if $(n, p(t)t^{-\alpha})$ is any element of $\mathbf{Z} \times \mathbf{Z}[t^{\pm 1}]$, then

$$\begin{aligned}\varphi(n, p(t)t^{-\alpha}) &= \varphi(n, 0) + \varphi(0, p(t))\varphi(0, t^{-\alpha}) \\ &= n\varphi(1, 0) + p(\varphi(0, t))\varphi(0, t)^{-\alpha}\varphi(0, 1) \\ &= n(1 - \varphi(0, 1)) + p(\varphi(0, t))\varphi(0, t)^{-\alpha}\varphi(0, 1).\end{aligned}$$

Thus φ is indeed uniquely determined once the value of $\varphi(0, t)$ is fixed, which proves that γ is an epimorphism in the category of rings.

- Here is another proof that γ is an epimorphism. Consider the diagram

$$\begin{tikzcd} \mathbf{Z}[t] \arrow[r, "\gamma"] \arrow[d, "\gamma"] & \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}] \arrow[d, "\gamma'"] \arrow[ddr, "\varphi"] & \\ \mathbf{Z} \times \mathbf{Z}[t^{\pm 1}] \arrow[r, "\gamma''"] \arrow[drr, "\psi"'] & (\mathbf{Z} \times \mathbf{Z}[t^{\pm 1}]) \otimes_{\mathbf{Z}[t]} (\mathbf{Z} \times \mathbf{Z}[t^{\pm 1}]) \arrow[dr, dotted, "\delta"] & \\ & & A \end{tikzcd}$$

By assumption the outer quadrangle commutes, which gives a unique map δ making the diagram commute. However, as was discussed in the proof of Lemma 1.4.9, the map γ' (and also γ'') is an isomorphism, so that $\varphi = \psi$.

Solution of Exercise 1.8.10: In general, a subset $S \subset X := \operatorname{Spec} A$ is stable under specialization iff its complement $X \backslash S$ is stable under generization. For example we show "$\Rightarrow$": if $x \in X \backslash S$ and $y \rightsquigarrow x$ are given, suppose $y \in S$. Then we would have $x \in S$ by the assumption. We also recall from Exercise 1.8.11 that $X \backslash S$ is constructible iff S is constructible.

We prove part (2) of th exercise only. Putting $V := X \backslash S$, we have to prove a constructible subset $V \subset X$ is closed iff it is stable under specialization. By Exercise 1.8.9, V is the image of a finite type map $\operatorname{Spec} B \to \operatorname{Spec} A$. It is closed by Lemma 1.7.19.

Solution of Exercise 1.8.13: This proof is due to Moret-Bailly https://mathoverflow.net/q/481465. Let Z be the image of $D(b)$ under $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$. For a prime $\mathfrak{p} \subset A$ we have $\mathfrak{p} \notin Z$ iff $\varnothing = D(b) \cap \varphi^{-1}(\mathfrak{p}) \overset{\text{Lemma 1.2.6}}{=} D(b) \cap \operatorname{Spec} B \otimes_A k(\mathfrak{p}) = \operatorname{Spec} B[b^{-1}] \otimes_A k(\mathfrak{p})$, which in turn is equivalent to b being nilpotent in $B \otimes_A k(\mathfrak{p})$. Given that B is free of rank d over A, the latter is a d-dimensional $k(\mathfrak{p})$-vector space. So being nilpotent is equivalent to $b^d = 0$. In the given basis $b^d = (a_1, \dots, a_d)$, so this amounts to $a_i = 0 \in k(\mathfrak{p})$, or equivalently $a_i \in \mathfrak{p}$. Therefore $\operatorname{Spec} A \backslash \varphi(Z) = \bigcap_{i=1}^d V(a_i)$.

Solution of Exercise 1.8.14: As in the proof before, for a prime $\mathfrak{p} \subset A$ we have $\mathfrak{p} \in \varphi(D(f))$ iff f is not nilpotent in $k(\mathfrak{p})[t]$. For a general ring A, a polynomial $f = \sum a_n t^n$ is nilpotent in $A[t]$ iff all its coefficients a_i are nilpotent in A, as one sees by induction on $\deg f$. For$A = k(\mathfrak{p})$, this means that $f = 0$. Thus Thus

$$\varphi(D(f)) = \{\mathfrak{p}, a_i \notin \mathfrak{p} \text{ for some } i\} = \bigcup_i D(a_i).$$

Solution of Exercise 1.8.15: : "$\Rightarrow$": first, $V(f)$ is clopen for any $f \in A$. This holds by (1.4.11). This implies that $V(I)$, for any finitely generated ideal $I \subset A$, is clopen. This then implies that any constructible subset is clopen.

"$\Leftarrow$": we use that $\operatorname{Spec} B$ is compact Hausdorff (Lemma 1.4.9(4)), and therefore any open subset is a *finite* union of fundamental open subsets $D(f)$.

Solution of Exercise 1.9.6: If k is no longer algebraically closed, there are prime ideals not of this form, e.g. in $\mathbf{R}[x]$ we have a prime ideal (x^2+1), whose residue field is $\mathbf{C}$.

4.2 Schemes

Solution of Exercise 2.8.3: By definition, X is covered by finitely many open affines $\operatorname{Spec} A$ with A being a finitely generated $\mathbf{Z}$-algebra. By Chevalley's theorem (Theorem 1.8.5), the image of $\operatorname{Spec} A$ in $\operatorname{Spec} \mathbf{Z}$ is constructible, hence so is the image of X. If $f(X)$ contains infinitely many closed points, it must contain the generic point as well by Definition 1.8.3.

Solution of Exercise 2.9.20: We have the implications "proper" $\Rightarrow$ "finite" (Proposition 2.9.12) and (by definition) "proper" $\Rightarrow$ "separated" and "finite" $\Rightarrow$ "separated" (Example 2.9.2). The converse implications do not hold: $\mathbf{A}^1 \to \operatorname{Spec} \mathbf{Z}$ is separated but not finite nor proper. The map $\mathbf{P}^1 \to \operatorname{Spec} \mathbf{Z}$ is proper, but not affine and in particular not finite (cf. the discussion after Definition 2.5.1).

Solution of Exercise 2.9.23: Throughout we use that the diagonal map Δ corresponds to the multiplication map $B \otimes_A B \to B$.

The map $A \to A[f^{-1}]$ is flat (being a localization) and nice, since $A[f^{-1}] \otimes_A A[f^{-1}] \to A[f^{-1}]$ is an isomorphism.

The map $\mathbf{Z} \to \mathbf{F}_p$ is not flat (since p is a zero-divisor in $\mathbf{F}_p$ but not in $\mathbf{Z}$), but nice: $\mathbf{F}_p \otimes_{\mathbf{Z}} \mathbf{F}_p \to \mathbf{F}_p$ is an isomorphism.

The map $\mathbf{Z} \to \mathbf{Z}[t]$ is flat (even free as a $\mathbf{Z}$-module), but not nice ($\mathbf{Z}[t] \otimes_{\mathbf{Z}} \mathbf{Z}[t] \to \mathbf{Z}[t]$ is surjective but not flat, and therefore not an open embedding).

We have a ring isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C}[t]/t^2+1 = \mathbf{C} \oplus \mathbf{C}$, and the multiplication map to $\mathbf{C}$ is given by $(z,w) \mapsto iz + \bar{i}w$. There is a splitting $\mathbf{C} \to \mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ given by $u \mapsto (\frac{1}{i}u, 0)$, so the map is an open embedding after passing to spectra. The map is flat (anything over a field is flat).

The composite of nice maps is nice, as one sees by expressing $\Delta_{g\circ f}$ as the composite of Δ_f and a pullback of Δ_g, cf. the proof of Lemma 2.9.7.

4.3 Cohomology of quasi-coherent sheaves

Solution of Exercise 3.5.13:

(1): We have

$$P_{\mathcal{O}_X}(m) = \dim_k \mathrm{H}^0(\mathbf{P}^n_k, \mathcal{O}(m)) = \dim k[t_0, \dots, t_n]_m$$

so that

$$P_{\mathcal{O}_X}(m) = \begin{cases} \binom{m+n}{n} = \frac{(m+n)(m+n-1)\cdots(m+1)}{n!} & m \geqslant 0 \\ 0 & m < 0 \end{cases}$$

We note that for $m \geqslant 0$, the above expression is a polynomial (in m) of degree n.

We have $P_{\mathcal{O}_X(e)}(m) = P_{\mathcal{O}_X}(e+m)$, which for any m with $e+m \geqslant 0$ agrees with the above polynomial. In particular, this is a numerical polynomial, with $\deg P_{\mathcal{O}_X(e)} = n$.

(2) In order to compute $P_{i_* \mathcal{O}_Y}$, we use the exact sequence

$$0 \to \mathcal{O}_X(-d) \xrightarrow{f} \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0.$$

The Euler characteristic is additive, so we get

$$\chi(i_*\mathcal{O}_Y(e)) = \chi(\mathcal{O}_X(e)) - \chi(\mathcal{O}_X(-d_1+e)).$$

Using (3.5.14), we may replace $\chi(F)$ by P_F throughout, provided that $e \gg 0$. We get

$$\begin{aligned} P_{i_*\mathcal{O}_Y}(e) &= P_{\mathcal{O}_X}(e) - P_{\mathcal{O}_X}(e-d_1) \\ &= \binom{e+n}{n} - \binom{e-d_1+n}{n} \\ &= \frac{(e+n)\cdots(e+1) - (e-d_1+n)\cdots(e-d_1)}{n!} \end{aligned}$$

In the enumerator, there is no term of order e^n (these cancel), so the degree is $\leqslant n-1$. If $d=0$ and $f=1$, we have $i_*\mathcal{O}_Y = 0$, so that $\deg P_{i_*\mathcal{O}_Y} = 0 < n-1$.

(3): We need to show that for each F there is some e_0 such that $\mathrm{H}^m(X, F(e)) = 0$ for all $m \geqslant 1$ and $e \geqslant e_0$. For any $F \in \mathrm{Coh}(X)$, we have an exact sequence

$$0 \to K \to \bigoplus \mathcal{O}(-d_i) \to F \to 0,$$

and we know $\mathrm{H}^m(X, -) = 0$ for any $m > n+1$. We do a downward induction on m. We have the exact sequence

$$\mathrm{H}^m(X, K(e)) \to \mathrm{H}^m(X, \bigoplus \mathcal{O}(e-d_i)) \to \mathrm{H}^m(X, F(e)) \to \mathrm{H}^{m+1}(X, K(e+1)),$$

and by induction the right hand term vanishes for $e \gg 0$, and this is also true for the second term (again for $e \gg 0$).

(4): We use the exact Koszul sequence

$$0 \to \underbrace{\mathcal{O}_X \to \ldots \to \bigoplus_i \mathcal{O}_X(-d_i) \to \mathcal{O}_X}_{=K(f_1,\ldots,f_m)} \to i_*\mathcal{O}_Y \to 0.$$

We see that the Euler characteristic of $i_*\mathcal{O}_Y$ only depends on the d_i, so we may assume $f_1 = t_0^{d_1}$, say. We have a short exact sequence of chain complexes

$$0 \to K(f_2, \ldots, f_m)(-d_1) \xrightarrow{f_1} K(f_2, \ldots, f_m) \to K(f_1, \ldots, f_m) \to 0.$$

Let us write $i' : Y' := V(f_2, \ldots, f_m) \subset X$ and P' for the numerical polynomial $P_{i'_*\mathcal{O}_{Y'}}$. We know by induction, beginning with the case $m=1$ discussed above, $\deg P' \leqslant n-m+1$. We obtain

$$\chi(X, i_*\mathcal{O}_Y(e)) = P'(e-d_1) - P'(e),$$

which is a polynomial of degree $\leqslant \deg P' - 1$.

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Index