Category theory

Jakob Scholbach

January 27, 2022

Contents

Chapter 0

Preface

These are growing notes for a lecture on category theory offered in Fall 2021 at the University of Münster. While category theory has practically no strict prerequisites, it is hard to appreciate category theory without some applications in other areas of mathematics. We will therefore include examples from algebra, topology, functional analysis and logic. Some very basic knowledge of some of these areas is therefore helpful.

- The exclamation mark (!) indicates that you should repeat some aspects of a definition etc. in order to make sure you are following the lecture.
- Three dots indicate some content that is not written in the notes, but will be explained in the lecture.
- These are links to video recordings of the lecture. 04.11.

Chapter 1

Introduction

Category theory offers a high-level abstraction for mathematical reasoning. It is comparable to the step of studying general linear equations

$$ax + by = c$$

instead of studying these one-by-one for specific choices of coefficients a, b and c. The process of abstracting the solution process for such an equation from the concrete choice of coefficients helps in understanding the essential points. A next step of abstraction is the concept of a vector space, which realizes the insight that the choice of a coordinate system does, for many purposes, not matter. Linear algebra is the mathematical theory that studies the totality of all vector spaces, and how they interact with each other. Category theory, in turn, studies all mathematical theories at a time. It seeks to identify what is common in different theories. For example, the concept of finiteness is a recurrent theme in algebra and topology. Finitely generated algebras, finitely generated or presented modules are pervasive in algebra, while compactness is an important condition in topology. Category theory offers a way to jointly address both conditions. While it is unsurprising that both are "in some way" a finiteness condition, it is not a priori obvious how to make sense of that notion since it requires to avoid talking about a set of generators of some algebra, or a finite subcovering of a given open covering. Category theory also clarifies how the relation between compact Hausdorff spaces and sets is similar to the relation between, say, $\mathbf{Z}[t]$-modules and abelian groups. Again, it does (and has to do) this by offering a framework to express the common features of both situation without specifically talking about topology, nor modules. This way, it helps clarify our thinking about these notions within one theory, and also helps to establish crucial bonds between different branches of mathematics.

Category theory, as exposed in this lecture, is only the penultimate step in this ladder of abstraction: ∞-categories and their theory, now central in branches such as contemporary algebraic topology or algebraic geometry, are not touched upon in this course. However, the concepts learned in this course carry over, often without any change in the statement, to ∞-category theory. In this sense, this course can serve as a preparation to go that one step further ...

Chapter 2

Basic notions

2.1 Categories

Category theory seeks to identify, clarify and exploit recurring patterns that occur throughout different branches of mathematics. This chapter provides the basic language needed to identify the common sources of such distant phenomena. A category is, roughly speaking, the arena in which you can perform some mathematical discipline.

Definition 2.1. A *category* is the datum consisting of a

- class $\mathrm{Obj}(C)$ called the *objects* of the category,
- for each pair $X, Y \in \mathrm{Obj}(C)$, a set $\mathrm{Hom}_C(X, Y)$ called the set of *morphisms* (from X to Y),
- for each $X \in \mathrm{Obj}(C)$ a distinguished element $\mathrm{id}_X \in \mathrm{Hom}_C(X, X)$ called the *identity morphism*,
- for each triple $X, Y, Z \in \mathrm{Obj}(C)$, a map called *composition*

$$\mathrm{Hom}_C(X, Y) \times \mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z), (f, g) \mapsto g \circ f,$$

such that the following conditions hold:

- for all $f \in \mathrm{Hom}_C(X, Y)$, $\mathrm{id}_Y \circ f = f$, $f \circ \mathrm{id}_X = f$,
- composition is associative, i.e., for any objects X, Y, Z, W, $f \in \mathrm{Hom}_C(X, Y)$, $g \in \mathrm{Hom}_C(Y, Z)$, $h \in \mathrm{Hom}_C(Z, W)$ we have
$$h \circ (g \circ f) = (h \circ g) \circ f$$
.

Remark 2.2. We will also write $\mathrm{Hom}(X, Y) := \mathrm{Hom}_C(X, Y)$. Another common notation is $C(X, Y)$. We also use $\mathrm{End}(X) := \mathrm{End}_C(X) := \mathrm{Hom}_C(X, X)$. Morphisms $f \in \mathrm{Hom}_C(X, Y)$ are also denoted as $X \xrightarrow{f} Y$.

Example 2.3. We have the following examples of categories. The assertion that the indicated objects morphisms form a category requires, in each case, an argument: saying that Top forms a category contains, in particular, the fact that the composite of two continuous maps is again continuous.

	objects	morphisms $\mathrm{Hom}(X, Y)$
Set	all sets	all maps from X to Y
Grp	all groups	all group homomorphisms from X to Y
Top	all topological spaces	all continuous maps from X to Y
Ban	all Banach spaces (over **C**, say)	all continuous **C**-linear maps
CRing	all commutative rings	all ring homomorphisms

Definition 2.4. A category D is called a *subcategory* if $\mathrm{Obj}(D) \subset \mathrm{Obj}(C)$ and if for any $X, Y \in D$, $\mathrm{Hom}_D(X, Y)$ is a subset of of $\mathrm{Hom}_C(X, Y)$ that contains id_X for all $X \in D$ and is stable under the composition.

It is called a *full subcategory*, if $\mathrm{Hom}_D(X, Y) = \mathrm{Hom}_C(X, Y)$. We will write

$$D \subset C$$

to indicate that D is a full subcategory.

The formation of full subcategories is very common; it amounts to selecting an arbitrary subclass of objects, while keeping the morphisms between them.

Example 2.5. By definition, we have the following (full) subcategories:

- $\mathrm{Ab} \subset \mathrm{Grp}$ is the category of abelian groups and all group homomorphisms, i.e., by definition a full subcategory of Grp
- Grp is a full subcategory of Mon, the category of monoids and monoid homomorphisms,
- $\mathrm{Ban}_{\leqslant 1}$ is the category of all Banach spaces with *contracting maps*, i.e.,

  $$\mathrm{Hom}_{\mathrm{Ban}_{\leqslant 1}}(V, W) := \{f \in \mathrm{Hom}_{\mathrm{Ban}}(V, W), |f| \leqslant 1\}$$

  (i.e., $|f(v)| \leqslant |v|$ for all $v \in V$). Note this indeed is a category, and it is a (non-full) subcategory of Ban. We will see later (e.g., in Exercise 4.9) that the two categories Ban and $\mathrm{Ban}_{\leqslant 1}$, while having the same objects, are profoundly different. This highlights the importance of the morphisms. However, in many situations the choice of morphisms (and more so, for identity morphisms and composites) is left implicit.
- $\mathrm{Haus} \subset \mathrm{Top}$ is the category of Hausdorff spaces, again by definition a full subcategory.

Definition 2.6. A category is called *small* if $\mathrm{Obj}(C)$ is a set (as opposed to a class).

Example 2.7. The above examples are not small, e.g., since all sets form a proper class. Small categories can be constructed by imposing some size condition. For example, fixing a set X, the subsets of X (and all maps between them) forms a small category.

Remark 2.8. In order to emphasize that the morphisms between two fixed objects form a set some authors also refer to a category as defined above as a *locally small* category. It is possible to drop this condition as well, so allowing $\mathrm{Hom}_C(X, Y)$ to be a class. Such categories are then referred to as *big* categories.

A set-theoretically clean method of incorporating these different cases is by working with *universes*, in which case one would choose two universes $U \subset V$. A small U-category C would then be such that $\mathrm{Obj}(C) \in U$ and all $\mathrm{Hom}_C(X, Y) \in U$. A locally small U-V-category C has $\mathrm{Obj}(C) \in V$ and all $\mathrm{Hom}_C(X, Y) \in U$. In this language, a big category would be just a small category with respect to the larger universe V.

As long as category theory is only used as a device to organize one's thinking, such set-theoretic issues can largely be ignored. The distinction between sets and classes of objects will, however, matter strongly in the adjoint functor theorem.

Example 2.9. Any *preordered set*[1] $(A, \leqslant)$ gives rise to a small category whose objects are the elements of A, and where $\mathrm{Hom}(a, b) = \{\star\}$ if $a \leqslant b$ and $\mathrm{Hom}(a, b) = \varnothing$ otherwise. Conversely, any small category C in which $\mathrm{Hom}_C(X, Y)$ has at most one element gives rise to a preordered set.

Definition 2.10. If C is a category, then the *opposite category* C^{op} is the category whose objects are $\mathrm{Obj}(C^{\mathrm{op}}) := \mathrm{Obj}(C)$, but for $X, Y \in \mathrm{Obj}(C^{\mathrm{op}})$ (i.e., $X, Y \in \mathrm{Obj}(C)$)

$$\mathrm{Hom}_{C^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_C(Y, X).$$

The identity morphisms and composition are inherited from C.

[1]Recall that a preordered set is defined by the following axioms: $a \leqslant a$ for all $a \in A$; $a \leqslant b$ and $b \leqslant c$ imply $a \leqslant c$.

This is a "stupid" definition in the sense that it just reverses the way we think of morphisms, and has otherwise no deeper content: it can be performed for any category C. Given a category, a meaningful description for C^{op} is often the starting point for interesting insights. E.g., parts of linear algebra can be viewed as a consequence of an equivalence of categories (see Definition 2.31) between the category of finite-dimensional vector spaces and its opposite category

$$(\mathrm{Vect}_k^{\mathrm{fd}})^{\mathrm{op}} \cong \mathrm{Vect}_k^{\mathrm{fd}}, V \mapsto V^* := \mathrm{Hom}(V, k).$$

The Hahn-Banach theorem in functional analysis is a (small) piece of a description of $\mathrm{Ban}^{\mathrm{op}}$ (Example 2.40).

2.2 Functors

The same way as differentiable maps are what makes differentiable manifolds talk to one another, or the same way as $\mathbf{Q}$-linear maps are what is needed to connect different $\mathbf{Q}$-vector spaces, category theory becomes useful mostly by connecting different categories. This is what functors do: they connect different categories and thus are the notion to use in order to connect different areas of mathematics.

Definition 2.11. Let C, D be two categories. A *functor* from C to D, commonly denoted as $F : C \to D$ is the following datum:

- a map (of classes), $F : \mathrm{Obj}(C) \to \mathrm{Obj}(D)$,
- for each pair of objects $X, Y \in \mathrm{Obj}(C)$, a map (of sets)

$$F : \mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(F(X), F(Y)).$$

To be a functor, the following conditions have to be satisfied:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for each $X \in \mathrm{Obj}(C)$,
- $F(g \circ f) = F(g) \circ F(f)$ for each $f \in \mathrm{Hom}_C(X, Y)$, $g \in \mathrm{Hom}_D(Y, Z)$.

Some authors call this a *covariant functor* and use the term *contravariant functor* for a map on the objects as above, and maps

$$\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(F(Y), F(X)$$

subject to similar conditions as above. In the notation of Definition 2.10, a contravariant functor is just a functor $C^{\mathrm{op}} \to D$.

Example 2.12. Consider the category C whose objects are $\mathbf{R}^n$ for $n \geqslant 0$, and such that

$$\mathrm{Hom}_C(\mathbf{R}^n, \mathbf{R}^m) = \{f : \mathbf{R}^n \to \mathbf{R}^m \text{ differentiable, } f(0) = 0\}$$

with the obvious composition, using that composites of differentiable functions are again differentiable. There is a *tangent space* functor

$$T : C \to \mathrm{Vect}_{\mathbf{R}}, \mathbf{R}^n \mapsto \mathbf{R}^n, f \mapsto T_0 f,$$

where $T_0 f$ (also sometimes denoted as $D_0 f$) is the total derivative. The functoriality of this just says

$$T_0(g \circ f) = T_0 g \circ T_0 f,$$

which is commonly known as the *chain rule*.

This example can be extended further by allowing not just $\mathbf{R}^n$ as objects, but arbitrary differentiable manifolds, and by considering the *tangent bundle*, so that the condition $f(0) = 0$ can be dropped.

Example 2.13. An important achievement in algebraic topology is the construction of a functor

$$\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$$

between the category of pointed topological spaces (and continuous maps preserving the base point) and the category of groups (and group homomorphisms). While detailing the construction of this functor is beyond the scope of these notes, let us at least indicate what this functor buys us:

- One computes, say, $\pi_1(S^1 \times S^1, *) = \mathbf{Z} \times \mathbf{Z}$ (the fundamental group of a torus), while
- $\pi_1(S^2, *) = 0$. (The hard work to be done by algebraic topologists lies in these computations.)
- By Lemma 2.20 below, since $\mathbf{Z} \times \mathbf{Z}$ is not isomorphic to the trivial group 0, this implies that the torus $S^1 \times S^1$ is *not* homeomorphic (i.e., isomorphic in Top) to the 2-sphere S^2.

Example 2.14. Given any category C, and any object $X \in C$, there are functors(!)

$$h_X : C^{\mathrm{op}} \to \mathrm{Set}, Y \mapsto \mathrm{Hom}_C(Y, X).$$

$$h^X : C \to \mathrm{Set}, Y \mapsto \mathrm{Hom}_C(X, Y).$$

The importance of these two functors cannot be overstated; we will study their relevance more deeply around the Yoneda lemma (§3.1) The former functor is called the *representing functor* attached to the object X, the latter the *corepresenting functor* attached to X. The two together can be combined to a functor

$$C^{\mathrm{op}} \times C \to \mathrm{Set}, (X, Y) \mapsto \mathrm{Hom}_C(X, Y).$$

Here the *product* of two categories C and D is defined by $\mathrm{Obj}(C \times D) = \mathrm{Obj}(C) \times \mathrm{Obj}(D)$ and

$$\mathrm{Hom}_{C \times D}((X, Y), (X', Y')) = \mathrm{Hom}_C(X, X') \times \mathrm{Hom}_D(Y, Y')$$

with the obvious identity and composition law.

Definition 2.15. A functor $F : C \to D$ is *faithful*, if all the maps (for $X, Y \in \mathrm{Obj}(C)$)

$$F : \mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_D(F(X), F(Y))$$

are injective. It is called *fully faithful*, if all these maps are bijective.

The above examples of categories may create the impression that a category is just some collection of sets, with certain restraints on the morphisms allowed. In the language of the definition, this amounts to having a (not necessarily fully) faithful functor

$$C \to \mathrm{Set}.$$

A category C is called *concretizable* if there is such a functor. Many, *but not all* categories are of that kind. The category HoTop of topological spaces up to homotopy is, by a non-trivial theorem of Freyd [Fre70], not concretizable. Even if a category is concretizable, the choice of such a functor U is not necessarily unique or natural: e.g., the two functors

$$U : \mathrm{Ban}_{\leqslant 1} \to \mathrm{Set}, V \mapsto V,$$

$$B_1 : \mathrm{Ban}_{\leqslant 1} \to \mathrm{Set}, V \mapsto B_1(V) := \{v \in V, |v| \leqslant 1\}$$

turn $\mathrm{Ban}_{\leqslant 1}$ into a concretizable category in two different ways. Thus, the role of an underlying set for objects in many categories usually takes the back seat in category theory.

A trivial observation is this:

Lemma 2.16. If $F : C \to D$ and $G : D \to E$ are two functors between categories C, D and E, the *composite*

$$G \circ F : C \to E$$

(defined on objects and morphisms in the obvious way) is again a functor.

2.3 Sameness

Definition 2.17. A morphism $f : X \to Y$ in a category C is called an *isomorphism* if there is another morphism $g : Y \to X$ (going in the other direction!, called an *inverse*) such that

$$f \circ g = \mathrm{id}_Y, g \circ f = \mathrm{id}_X.$$

One shows(!) that such an inverse is unique if it exists. It is therefore also denoted f^{-1}.

Example 2.18. A map in Set is an isomorphism iff it is bijective.

Example 2.19. In the category TopGp of topological groups (with continuous group homomorphisms), the logarithm

$$\log : \mathbf{R}^{>0} \to \mathbf{R}$$

is an isomorphism, since the exponential $\exp : \mathbf{R} \to \mathbf{R}^{>0}$ is again a continuous group homomorphism and the two composites are the respective identities.

Lemma 2.20. Given a functor $F : C \to D$ and an *iso*morphism $f : X \to Y$, the morphism

$$F(f) : F(X) \to F(Y)$$

is an isomorphism (in D).

We refer to this fact by saying that any functor *preserves* isomorphisms.

Proof. Indeed, $F(f^{-1}) : F(Y) \to F(X)$ satisfies

$$F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(\mathrm{id}_X) = \mathrm{id}_F(X)$$

and similarly with the other composite. (Note this uses both conditions on being a functor.) □

Conversely, though, it is not necessarily true that f is an isomorphism whenever $F(f)$ is. Applied to the forgetful functor

$$U : \mathrm{TopGp} \to \mathrm{Set}$$

this means that just checking log is bijective (so that $U(\log)$ is an isomorphism) is a priori not enough to ensure that its set-theoretically existing inverse is actually continuous and a group homomorphism. See Exercise 2.4 for further examples.

Definition 2.21. A functor $F : C \to D$ is called *conservative* if the following holds: a morphism f in C is an isomorphism if (and therefore iff) $F(f)$ is an isomorphism.

For example, the forgetful functor

$$U : \mathrm{Grp} \to \mathrm{Set}$$

is conservative: if a group homomorphism f is a bijection (i.e., $U(f)$ is an isomorphism), then the set-theoretical inverse f^{-1} is automatically a group homomorphism and is an inverse in Grp. Conservative functors are quite wide-spread. We will encounter them more systematically around the Barr–Beck monadicity theorem (§6).

Pausing briefly our discussion of sameness, we also introduce two weaker notions than isomorphism:

Definition 2.22. A morphism $f : X \to Y$ in a category C is called a

- *monomorphism* if for every two arrows $t_0, t_1 : T \to X$ with $f \circ t_0 = f \circ t_1$ we have $t_0 = t_1$. Equivalently, if $\mathrm{Hom}(T, X) \xrightarrow{f \circ -} \mathrm{Hom}(T, Y)$ is injective for any $T \in C$.
- *epimorphism* if it is a monomorphism in the opposite category C^{op}, or if for every two arrows $t_0, t_1 : Y \to T$ such that $t_0 \circ f = t_1 \circ f$ we have $t_0 = t_1$.

Clearly(!) every isomorphism is both a monomorphism and an epimorphism. The converse holds in the category of Set, but does *not* hold in other categories such as CRing, see Exercise 2.8.

Working towards the notion of sameness in category theory, it is at first suggestive to consider the following idea:

Definition 2.23. A functor $F : C \to D$ is an *isomorphism of categories* if there is another functor $G : D \to C$ such that the two composites $G \circ F = \mathrm{id}_C$ and $F \circ G = \mathrm{id}_D$ are the respective identity functors, i.e., for each object $G(F(X)) = X$, $F(G(Y)) = Y$ and likewise for each morphism.

Example 2.24. Let $k \subset K$ be a finite Galois extension.[2] Let $G := \mathrm{Gal}(K/k) := \{\sigma : K \to K, \sigma|_k = \mathrm{id}_k\}$ be the group of field automorphisms of K that fix k elementwise.

The *main theorem of Galois theory* can be stated as a bijection

$$\{k \subset L \subset K\} \rightleftarrows \{H \subset G\}$$

between the intermediate fields and the subgroups of G. The maps are given by

$$L \mapsto \mathrm{Gal}(K/L) := \{\sigma : K \to K, \sigma|_L = \mathrm{id}_L\}(\subset G)$$

and conversely

$$H \mapsto \mathrm{Fix}(H) := \{x \in K, \sigma x = x \quad \forall \sigma \in H\}(\subset K).$$

That is we have identities:

$$L = \mathrm{Fix}(\mathrm{Gal}(K/L)), H = \mathrm{Gal}(K/\mathrm{Fix}(H)).$$

Looking slightly closer, one notices that to an inclusion $L \subset L'$ corresponds an inclusion (note the direction changes) $\mathrm{Gal}(K/L') \subset \mathrm{Gal}(K/L)$ and conversely to $H \subset H'(\subset G)$ corresponds $(k \subset)\mathrm{Fix}(H') \subset \mathrm{Fix}(H)(\subset K)$.

We can therefore add this extra structure and consider the following categories

- Sub_G: objects are the subgroups of G, with morphisms given by inclusions.
- Fields_k^K: objects are subfields $k \subset L \subset K$ and morphisms are inclusions $L \subset L'(\subset K)$.

Note that these categories are (categories associated to) posets (Example 2.9), since there is at most one morphism between any two objects. Thus, the bijection on the level of objects and the functoriality (i.e., preservation of inclusions) gives automatically rise to an *isomorphism* of categories

$$\mathrm{Gal}(K/-) : \mathrm{Fields}_k^K \rightleftarrows \mathrm{Sub}_G^{\mathrm{op}} : \mathrm{Fix}(-).$$

It is possible to slightly enlarge the morphisms allowed by consider the category of transitive G-sets (these are of the form G/H for some subgroup H) instead of Sub_G and on the other side allow not just inclusions, but arbitrary field homomorphisms $L \to L'$ (whose restriction to k is the identity). In this case, one still obtains an isomorphism of categories (which are no longer posets), see [Rie17, Example 1.3.15] for a precise statement.

Keeping this example in mind, let us however point out that isomorphisms of categories are rare in practice. What made it work for the example of Galois theory is that there are few (almost no in the first case) isomorphisms. This is, however, not so in many other situations, like the following example.

Example 2.25. Let k be a field and $\mathrm{Vect}_k^{\mathrm{fd}}$ the category of finite-dimensional k-vector spaces and k-linear maps. Consider the functors

$$(\mathrm{Vect}_k^{\mathrm{fd}})^{\mathrm{op}} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Vect}_k^{\mathrm{fd}}$$

defined on objects by $F(V) = \mathrm{Hom}(V, k)$ and $G(V) = \mathrm{Hom}(V, k)$ and on morphisms in the "obvious way".

[2] Recall a finite field extension $k \subset K$ is called *Galois* if it is separable and normal. Equivalently, the number of automorphisms $\sigma : K \to K$ with $\sigma|_k = \mathrm{id}_k$ equals $\dim_k K$. See, e.g., [**Bower:Category**] for a leisurely introduction to Galois theory with an eye towards its categorical interpretation.

As usual, the set $\mathrm{Hom}(V,k)$ is regarded as a k-vector space in its own right. (This is an example of a so-called *inner homomorphism*, i.e., a case where the set of homomorphisms happens to be an object in the category itself again.)

The composites are given by

$$V \mapsto G(F(V)) = \mathrm{Hom}(\mathrm{Hom}(V,k),k).$$

Linear algebra tells us that, for finite-dimensional vector spaces V, the natural map

$$V \to \mathrm{Hom}(\mathrm{Hom}(V,k),k), v \mapsto (f \mapsto f(v))$$

is an isomorphism, but the double dual is not the *same* vector space, so F and G do not constitute an isomorphism of categories.

However, from linear algebra we also know that demanding that two vector spaces are the same is actually too strong: for all purposes it is enough to ensure they are isomorphic. We want to relax the above definition by just requiring that $G(F(V))$ be isomorphic to V. However, these isomorphisms should not be chosen at will individually for each V, but should rather be in harmony with one another. In order to express this latter idea we need to introduce another concept:

Definition 2.26. Let $F, G : C \to D$ be two functors (going in the same direction). A *natural transformation*

$$\alpha : F \Rightarrow G$$

is the datum of a morphism $\alpha(X) : F(X) \to G(X)$ for all objects $X \in \mathrm{Obj}(C)$, such that for each morphism $f : X \to Y$ in C, the square

$$\begin{tikzcd} F(X) \arrow[r, "\alpha(X)"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha(Y)"] & G(Y) \end{tikzcd}$$

commutes, i.e., $\alpha(Y) \circ F(f) = G(f) \circ \alpha(X)$.

The natural transformation α is called a *natural isomorphism* if the maps $\alpha(X)$ are isomorphisms for all $X \in \mathrm{Obj}(C)$.

Example 2.27. Recall the category $\mathrm{Ban}_{\leqslant 1}$ of Banach spaces and contractive maps and the forgetful and "ball of radius 1" functors:

$$U : \mathrm{Ban}_{\leqslant 1} \to \mathrm{Set}, V \mapsto V,$$

$$B_1 : \mathrm{Ban}_{\leqslant 1} \to \mathrm{Set}, V \mapsto B_1(V) := \{v \in V, |v| \leqslant 1\}$$

There is a natural transformation

$$B_1 \Rightarrow U$$

given by the inclusions $B_1(V) \subset V$ (which trivially commute with any map $f : V \to W$). Of course it is not a natural isomorphism since the inclusions are never (except for $V = 0$) bijections.

For each object $V \in \mathrm{Ban}_{\leqslant 1}$, we can also contemplate the map $\beta(V) : V \to B_1(V), v \mapsto \frac{v}{\max(1,|v|)}$. This does *not* give rise to a natural transformation

$$U \Rightarrow B_1,$$

since for a linear contractive map $f : V \to W$, we will not in general have

$$\beta(f(v)) = f(\beta(v)).$$

Example 2.28. The contravariant *power set functor*

$$P : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$$

is given on objects by $A \mapsto P(A) := \{\text{the subsets of } A\}$ and for morphisms $f : A \to B$ given by $P(f) := f^{-1} : V \subset B \mapsto f^{-1}(V) \subset A$.

Another functor is the representing functor

$$h_2 := \mathrm{Hom}(-, 2) : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}, A \mapsto \mathrm{Hom}(A, 2)$$

where $2 = \{0, 1\}$ is a two-element set. There is a natural isomorphism

$$\mathrm{Hom}(-, 2) \Rightarrow P$$

given on objects by $\mathrm{Hom}(A, 2) \ni f \mapsto (f^{-1}(0) \subset A) \in P(A)$.

We introduce some language to conveniently talk about such examples: the above example then says that the functor P is representable by 2.

Definition 2.29. A functor $F : C^{\mathrm{op}} \to \mathrm{Set}$ is called *representable* if there is a natural isomorphism

$$F \cong h_X$$

for some object $X \in C$. Dually, a functor $F : C \to \mathrm{Set}$ is called *corepresentable* if there is an isomorphism $F \cong h^X$. Such an object is called a *(co)representing object*.

Remark 2.30.

- Further elementary examples are discussed in Exercise 2.7.
- Once we discuss the Yoneda lemma, we will see that given another object X' and an isomorphism $F \cong h_{X'}$ there is an isomorphism $X \cong X'$ (see Corollary 3.7). Moreover, if we demand the natural compatibility condition with the given isomorphisms, such an isomorphism is unique.
- The notion of representable functors arose in algebraic topology where one constructs so-called *Eilenberg-Maclane spaces* $H(\mathbf{Z}, n) \in \mathrm{Top}$. For example, $H(\mathbf{Z}, 1) = S^1$. One then shows that singular cohomology
$$\mathrm{H}^n(X, \mathbf{Z}) = \mathrm{Hom}_{\mathrm{HoTop}}(X, H(\mathbf{Z}, n)),$$
so that cohomology is a representable functor on the category of topological spaces up to homotopy. See, e.g., [**Hatcher:Algebraic**].

Here comes the notion of sameness that is much more useful than the isomorphism of categories.

Definition 2.31. A pair of functors $F : C \to D$ and $G : D \to C$, also pictured as

$$C \underset{G}{\overset{F}{\rightleftarrows}} D$$

is called an *equivalence of categories*, if there are natural isomorphisms

$$\mathrm{id}_C \Rightarrow G \circ F, \ \mathrm{id}_D \Rightarrow F \circ G.$$

Equivalences of categories will be denoted by $\cong$. We say that $F : C \to D$ is an equivalence if there is a functor G satisfying the above conditions.

Remark 2.32. Contrast this with the notion of an isomorphism of categories, in which case we are demanding *equalities*

$$\mathrm{id}_C = G \circ F, \ \mathrm{id}_D = F \circ G.$$

Example 2.33. The functors in Example 2.25 constitute an equivalence of categories: the maps $V \to G(F(V))$ and $W \to F(G(W)) = \mathrm{Hom}(\mathrm{Hom}(W, k), k)$ are functorial in V resp. W. Moreover, they are isomorphisms for finite-dimensional vector spaces.

There is a useful criterion to show that a given functor $F : C \to D$ is part of an equivalence of categories. To state it we introduce some more language:

Definition 2.34. A functor $F : C \to D$ is *essentially surjective* if for each $Y \in D$ there is some $X \in C$ and an isomorphism $F(X) \cong Y$.

Proposition 2.35. Let $F : C \to D$ be a functor. The following are equivalent:
(1) F is part of an equivalence of categories (i.e., there is a functor $G : D \to C$ such that the conditions in Definition 2.31 are satisfied),
(2) F is fully faithful (Definition 2.15) and essentially surjective.

Proof. The direction (1) $\Rightarrow$ (2) is left as an exercise. Conversely, assume F is essentially surjective and fully faithful. Using the axiom of choice (for classes), we can choose for each $Y \in D$ some object, denoted $G(Y) \in C$ such that $F(G(Y)) \cong Y$. To complete the definition of G, we specify it on morphisms such that this (somewhat arbitrary choice) becomes functorial. Let $f : X \to Y$ be a morphism in D. Consider the composite

$$F(G(X)) \cong X \xrightarrow{f} Y \cong F(G(Y)).$$

Since F is fully faithful, there is exactly one morphism $G(X) \to G(Y)$ that is mapped to this composite under F. Define $G(f)$ to be that map. In other words, the diagram

$$\begin{tikzcd} F(G(X)) \arrow[r, "\cong"] \arrow[d, "F(G(f))"'] & X \arrow[d, "f"] \\ F(G(Y)) \arrow[r, "\cong"] & Y \end{tikzcd}$$

commutes. One checks the conditions for G being a functor(!) . By the diagram above, we have a (functorial) isomorphism $F \circ G \cong \mathrm{id}_D$. We conclude

$$F \circ (G \circ F) = (F \circ G) \circ F \cong \mathrm{id}_D \circ F = F \circ \mathrm{id}_C.$$

One checks, using the full faithfulness of F, that this implies $G \circ F \cong \mathrm{id}_C$. □

Example 2.36. Given a category C with a full subcategory $C' \subset C$ such that the inclusion is essentially surjective, then $C' \cong C$.

Given a category C', we can thus for example consider a category C with $\mathrm{Obj}(C) = \mathrm{Obj}(C') \sqcup \mathrm{Obj}(C')$ (two copies of objects of C'), and

$$\mathrm{Hom}_C(X, Y) = \mathrm{Hom}_{C'}(X, Y).$$

For $X \in C'$, writing X_1 for the object regarded in the first copy of C', and X_2 for the one in the second copy, the identity id_X in particular gives a morphism

$$\iota_X \in \mathrm{Hom}_C(X_1, X_2)(= \mathrm{Hom}_{C'}(X, X)).$$

This map is an isomorphism(!) . Thus the functor $C' \to C$ mapping $X \mapsto X_1$ and $f \mapsto f$ is fully faithful and essentially surjective. It is therefore an equivalence of categories. In a sense, C consists of another copy of C', with just different names for the objects.

Example 2.37. Let k be a field and Mat_k the category whose objects are the natural numbers $\mathbf{N} = \{0, 1, \dots\}$ and $\mathrm{Hom}_{\mathrm{Mat}_k}(n, m) = k^{n \times m}$ with composition given by matrix multiplication. The obvious functor

$$F : \mathrm{Mat}_k \to \mathrm{Vect}_k^{\mathrm{fd}}, n \mapsto k^n$$

and on morphisms (for $A \in k^{n \times m}$) $F(A) : k^n \to k^m, x \mapsto Ax$ is an equivalence of categories by the preceding proposition. Note that defining an inverse functor requires choices $k^n \cong V$ for all $V \in \mathrm{Vect}_k^{\mathrm{fd}}$, i.e., the choice of bases of all vector spaces.

Corollary 2.38. There is an equivalence of categories

$$(\mathrm{Vect}_k^{\mathrm{fd}})^{\mathrm{op}} \xrightarrow{\cong} \mathrm{Vect}_k^{\mathrm{fd}}, V \mapsto V^* := \mathrm{Hom}(V, k).$$

Proof. This follows from the preceding example, observing that

$$\mathrm{Mat}_k^{\mathrm{op}} \cong \mathrm{Mat}_k, n(\in \mathbf{N}) \mapsto n, \mathrm{Hom}_{\mathrm{Mat}_k^{\mathrm{op}}}(n,m) = k^{m\times n} \xrightarrow{\cong} k^{n\times m} = \mathrm{Hom}_{\mathrm{Mat}_k}(n,m),$$

where at the right we use the natural isomorphism $k^{m\times n} \cong k^{n\times m}$ given by taking the transpose of a matrix. (Alternatively, it is also possible to prove this using Proposition 2.35 directly.) □

Definition 2.39. An equivalence of the form

$$C^{\mathrm{op}} \cong D$$

is called a *duality*. (Some authors, however, do not use the term duality in a precise technical sense.)

Example 2.40. In functional analysis there is the notion of *Smith spaces* (also known as *Waelbroeck spaces*).[3] There is an equivalence of categories (see, e.g., [**Scholze:Lectures**] for a recent exposition)

$$\mathrm{Hom}(-, \mathbf{R}) : \mathrm{Smith}^{\mathrm{op}} \cong \mathrm{Ban} : \mathrm{Hom}(-, \mathbf{R}).$$

The *Hahn–Banach theorem* can be read off from this equivalence: it is commonly stated by saying that for an inclusion of Banach spaces and a map f, there is an extension g with the same norm:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, hook] & \mathbf{R} \\ W. \arrow[ur, dotted, "g"] & \end{tikzcd}$$

One checks (quickly) that inclusions of Banach spaces correspond to surjections of the corresponding Smith spaces. In the context of Smith spaces, the corresponding statement is trivial: any map $a : \mathbf{R} \to A$ corresponds to just an element in A and this can always be lifted along a surjection:

$$\begin{tikzcd} & B \arrow[d, two heads] \\ \mathbf{R} \arrow[r, "a"] \arrow[ur, dotted, "b"] & A. \end{tikzcd}$$

2.4 Exercises

Exercise 2.1. Let M be a monoid (e.g., a group). Show that M gives rise to a category, denoted BM with a single object, denoted $\star$, and such that $\mathrm{Hom}_{BM}(\star, \star) = M$ with composition given by the multiplication in the monoid M.

Unwind the definition of a functor $BM \to \mathrm{Set}$ and explain why such a functor is called a set with an M-action (or M-set).

Exercise 2.2. A category in which every morphism is an isomorphism is called a *groupoid*. Explain why that terminology has been adopted.

Let C be a groupoid and $f : X \to Y$ be an (iso-)morphism. Show that f gives rise to a group isomorphism

$$\mathrm{End}_C(X) \to \mathrm{End}_C(Y).$$

Consider the groupoid C consisting of all finite-dimensional vector-spaces (over a fixed field k), where we only allow isomorphisms as morphisms. Let $V \in C$ and $f : k^n \to V$ an isomorphism. What is the relation of the above to the representation of linear maps $V \to V$ in terms of a basis?

[3] A Smith space is a complete locally convex topological $\mathbf{R}$-vector space V that admits a compact absolutely convex subset $K \subset V$ such that $V = \bigcup_{c>0} cK$ with the induced compactly generated topology on V.

Exercise 2.3. Recall that the *center* of a monoid M is defined as

$$Z(M) = \{g \in M | gh = hg \; \forall h \in M\}.$$

(1) Is the assignment

$$\mathrm{Mon} \to \mathrm{AbMon}, G \mapsto Z(G)$$

part of a functor (taking values in the category of abelian monoids and monoid homomorphisms)?

(2) Let $M\mathrm{Set}$ be the category of M-sets (Exercise 2.1). Construct a bijection

$$Z(M) = \mathrm{Hom}_{M \times M^{\mathrm{op}}-\mathrm{Set}}(M, M).$$

Exercise 2.4. Which of the following functors are faithful, which are fully faithful, which are conservative? Functors called U are the obvious forgetful functors.

(1) $U : \mathrm{Top} \to \mathrm{Set}$

(2) $U : \mathrm{CptHaus} \to \mathrm{Set}$ (compact Hausdorff spaces with continuous maps)

(3) $U : \mathrm{Mod}_{\mathbf{Q}} \to \mathrm{Mod}_{\mathbf{Z}}$. Here Mod_R denotes the category of R-modules, for a ring R, with morphisms being R-linear maps. Thus $\mathrm{Mod}_{\mathbf{Z}} = \mathrm{Ab}$ is the category of abeliang groups, and $\mathrm{Mod}_{\mathbf{Q}}$ is the category of $\mathbf{Q}$-vector spaces.

(4) $U : \mathrm{Ban}_{\leqslant 1} \to \mathrm{Ban}$

(5) $U : \mathrm{Ban} \to \mathrm{Vect}_{\mathbf{C}}$

(6) $- \otimes_{\mathbf{Z}} \mathbf{Q} : \mathrm{Mod}_{\mathbf{Z}} \to \mathrm{Mod}_{\mathbf{Q}}$.

Note: To decide (2) and (5), you need to use a fact from point-set topology, respectively functional analysis.

Exercise 2.5. Let $\mathrm{FinSet}^{\cong}$ be the category of finite sets, where we only consider bijections as morphisms. Define a functor $\mathrm{Sym} : \mathrm{FinSet}^{\cong} \to \mathrm{Set}$ by sending a finite set X to its set of permutations. Defne another functor $\mathrm{Ord} : \mathrm{FinSet}^{\cong} \to \mathrm{Set}$ by sending X to the set $\mathrm{Ord}(X)$ of total orderings on X, and a bijection $f : X \to Y$ to the map

$$\mathrm{Ord}(f) : \mathrm{Ord}(X) \to \mathrm{Ord}(Y)$$

that sends a total order $\leqslant_X$ on X to the following order $\leqslant_Y$ on Y:

$$y \leqslant_Y y' \Leftrightarrow f^{-1}(y) \leqslant_X f^{-1}(y').$$

Show that for each finite set X, $\mathrm{Sym}(X) \cong \mathrm{Ord}(X)$, but that there is no isomorphism of functors $\mathrm{Sym} \cong \mathrm{Ord}$. This is expressing the fact that any finite set can be totally ordered, but not functorially (or, as one also says, not canonically).

Exercise 2.6. Complete the proof of Proposition 2.35.

Exercise 2.7. Show that the functors (U denotes the usual forgetful functors)

$$U : \mathrm{Ring} \to \mathrm{Set}$$

$$U : \mathrm{Ab} \to \mathrm{Set}$$

$$B_{\leqslant 1} : \mathrm{Ban}_{\leqslant 1} \to \mathrm{Set}$$

$$\mathrm{Op} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}, X \mapsto \mathrm{Op}(X) := \{\text{the set of open subsets of } X\}, f \mapsto f^{-1}$$

are (co)representable.

Exercise 2.8. Show that a morphism in CRing is a monomorphism iff it is injective. Show that a morphism in CRing is an epimorphism if it is surjective. The converse is false: show that $\mathbf{Z} \to \mathbf{Q}$ is an epimorphism.

Note: the description of epimorphisms in CRing is subtle. A sample theorem is the following: suppose $f : A \to B$ is a local homomorphism between local (commutative) rings with A Noetherian. If f is an epimorphism then $B = C[S^{-1}]$ is a localization of an A-algebra C that is finitely generated [**Ferrand:Monomorphismes**]. The expectation that an epimorphism is surjective can be partly salvaged: surjective ring homomorphisms $A \to B$ are precisely the finite (i.e., B is a finitely generated A-module) epimorphisms [**StacksProject**].

Exercise 2.9. The following exercise shows that the condition that a naturality of a transformation is a strong condition.

Consider the functor

$$F : \mathrm{Set} \to \mathrm{Set}, X \mapsto X \times X, (X \xrightarrow{f} Y) \mapsto (X \times X \xrightarrow{f \times f} Y \times Y)$$

as well as the identity functor $\mathrm{id} : \mathrm{Set} \to \mathrm{Set}$.

- Show that the only natural transformations $F \Rightarrow \mathrm{id}$ are given by the projections on the two coordinates, i.e., on objects by $X \times X \to X$, $(x_0, x_1) \mapsto x_0$ or x_1.

  Hint: consider first what happens for $X = \{0, 1\}$.

- Give an example of an "unnatural" transformation, i.e., a collection of maps $F(X) \to X$ for all sets X, that fails to be natural.

Exercise 2.10. Show that a fully faithful functor is conservative.

Chapter 3

Functor categories

Given two categories C and D, the totality of all functors $C \to D$ carries the structure of a category in its own right.

Definition 3.1. Let $F, G, H : C \to D$ be three functors, and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ natural transformations. The composite

$$\beta \circ \alpha : F \Rightarrow H$$

is defined in the expected way, for objects $X, Y \in C$ and morphisms $f : X \to Y$:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha(X)"] \arrow[dd, bend right, "(\beta\circ\alpha)(X)"'] & F(Y) \arrow[d, "\alpha(Y)"] \arrow[dd, bend left, "(\beta\circ\alpha)(X)"] \\
G(X) \arrow[r, "G(f)"] \arrow[d, "\beta(X)"] & G(Y) \arrow[d, "\beta(Y)"] \\
H(X) \arrow[r, "H(f)"] & H(Y).
\end{tikzcd}$$

Note this is in contrast of the composition of *functors* (which would, given a functor $F : C \to D$ and another functor to another category, $F' : D \to E$, give a functor $F' \circ F : C \to E$.) It can be useful to illustrate compositions of natural transformations like so:

$$\begin{tikzcd}
C \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & D.
\end{tikzcd}$$

For a more comprehensive approach to visualizing category theory, see e.g., [**Marsden:Category**].

Definition and Lemma 3.2. Let C, D be two categories. We define the functor category

$$\mathrm{Fun}(C, D)$$

(also denoted as D^C) to have as objects the functors $C \to D$ and as morphisms the natural transformations between such functors and composition being the one in Definition 3.1).

If C is a small category and D is small, then this is again a small category. If C is a small category and D is locally small (i.e., $\mathrm{Hom}_D(-,-)$ are sets), then this is again locally small. (If no smallness condition is imposed on C, then $\mathrm{Fun}(C, D)$ will be a big category, cf. Remark 2.8.)

Proof. Verifying the conditions of a category is routine. About the size issues note that the morphisms (i.e., natural transformations) between two given objects (i.e., functors F and G) form a set, in fact a subset of $\prod_{X \in C} \mathrm{Hom}_D(F(X), G(X))$ (which is a set since each factor is one and the product is over a *set* of objects).

If, in addition D is small, then $\mathrm{Obj}(\mathrm{Fun}(C, D)) \subset \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Obj}(C), \mathrm{Obj}(D))$ is a set, i.e., the functor category is small. $\square$

3.1 The Yoneda embedding

Definition 3.3. Let C be a small category. A presheaf on C is a functor $C^{\mathrm{op}} \to \mathrm{Set}$. The category of *presheaves* is the category

$$\widehat{C} := \mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}).$$

The category of *copresheaves* is the category

$$\mathrm{CoPSh}(C) := \mathrm{Fun}(C, \mathrm{Set}).$$

Example 3.4. For a topological space X, consider the category $\mathrm{Op}(X)$ of open subsets, ordered by inclusion. Then a presheaf on X (in the sense of any topology textbook) is a presheaf (in the above sense) on $\mathrm{Op}(X)$. Concretely, a set $F(U)$ for each $U \subset X$ open, and for each inclusion $U \subset V$, a (so-called) *restriction map* $F(V) \to F(U)$ that is compatible with respect to further inclusions.

Fixing a "target" topological space T, a prototypical example is the representable presheaf

$$U \mapsto \mathrm{Hom}_{\mathrm{Top}}(U, T).$$

For example, for $T = \mathbf{R}$, this is just the presheaf that assigns to any U the continuous, real-valued functions on U.

Morphisms $F \to G$ of presheaves are collections of maps $F(U) \to G(U)$ that are compatible with the restriction maps.

Copresheaves are similar, except that there are maps $F(U) \to F(V)$ for $U \subset V$. These maps are called corestriction maps. Copresheaves are more rare than presheaves in practice. An example is the copresheaf of compactly supported functions:

$$U \mapsto \{f : U \to \mathbf{R}, \mathrm{supp}(f) \text{ is compact }\},$$

with the corestriction maps (for $U \subset V$) given by extending a function by zero.

Let C be any small category, $X \in C$ and $F : C^{\mathrm{op}} \to \mathrm{Set}$ a presheaf on C. There is a map (of sets)

$$F(X) \to \mathrm{Hom}_{\widehat{C}}(h_X, F)$$

that sends $f \in F(X)$ to the natural transformation $h_X \to F$ that is given on objects $Y \in C^{\mathrm{op}}$ by $h_X(Y) = \mathrm{Hom}_C(Y, X) \ni \alpha \mapsto F(\alpha)(f) \in F(Y)$. (Note here $F(\alpha) : F(X) \to F(Y)$ since F is contravariant. Note also that given a morphism $Y \to Z$ in C^{op}, i.e., a morphism $y : Z \to Y$ in C, the diagram

$$\begin{tikzcd}
h_X(Y) = \mathrm{Hom}_C(Y, X) \arrow[r] \arrow[d, "y^* = \mathrm{Hom}_C(y, X)"] & F(Y) \arrow[d, "F(y)"'] \\
h_X(Z) = \mathrm{Hom}_C(Z, X) \arrow[r] & F(Z)
\end{tikzcd}$$

commutes since F is a functor(!) Thus we have indeed defined an element in $\mathrm{Hom}_{\widehat{C}}(h_X, F)$.)

Conversely, there is a map (of sets)

$$\mathrm{Hom}_{\widehat{C}}(h_X, F) \to F(X)$$

that sends a natural transformation $g : h_X \to F$ to $g(\mathrm{id}_X) \in F(X)$ (note the evaluation of g at X gives a map $h_X(X) = \mathrm{Hom}_C(X, X) \ni \mathrm{id}_X \to g(\mathrm{id}_X) \in F(X)$).

Lemma 3.5. The above two maps are inverse to each other, so we have an isomorphism.

The (completely formal) proof is relegated to Exercise 3.2.

Lemma 3.6. Let C be any small category. The functor

$$C \to \widehat{C} := \mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}), X \mapsto h_X := \mathrm{Hom}_C(-, X)$$

is fully faithful. It is called the *Yoneda embedding*. It establishes an equivalence between C and the full subcategory of $\widehat{C}$ spanned by the representable functors.

Dually, there is a fully faithful functor

$$C^{\mathrm{op}} \to \mathrm{CoPSh}(C), X \mapsto h^X.$$

Proof. The full faithfulness follows directly from Lemma 3.5:

$$\mathrm{Hom}_C(X, Y) = h_Y(X) = \mathrm{Hom}_{\widehat{C}}(h_X, h_Y).$$

The last statement is then immediate from Proposition 2.35, since the representable functors are, by definition, the ones in the essential image of the Yoneda embedding. □

The above lemma, known as the Yoneda lemma is completely formal, yet very powerful. We will see later on (§8.3) that the formation

$$C \mapsto \widehat{C}$$

is, in a certain precise sense, comparable to taking the completion of a metric space. A related aspect is that, unlike an arbitrary category C, the category $\widehat{C}$ is large enough to do all reasonable operations, as we will see in Lemma 4.34. This process of enlarging the category can be useful to perform certain constructions, as is sketched in the more advanced Exercise 3.3. A more elementary example how functor categories can be useful occurs in Exercise 3.1.

Some immediate application:

Corollary 3.7. Let C be a category and $X, Y \in C$ two objects. There is a bijection

$$\mathrm{Hom}_C(X, Y) \cong \mathrm{Hom}_{\widehat{C}}(h_X, h_Y)$$

and under this bijection isomorphisms (of objects in C) correspond to isomorphisms (of functors). In particular, if $X \cong Y$ iff $h_X \cong h_Y$ (isomorphism of functors).

In particular, let $F : C^{\mathrm{op}} \to \mathrm{Set}$ be a representable functor. If X and $Y \in C$ are two representing objects for F, they are isomorphic.

Example 3.8. Consider a polynomial in two variables, with coefficients $a_{i,j} \in \mathbf{Z}$:

$$f(x, y) = \sum a_{i,j} x^i y^j \in \mathbf{Z}[x, y].$$

Such a polynomial (and, exactly the same way, a polynomial in more variables, or several polynomials in several variables) gives rise to a functor

$$\mathrm{CRing} \to \mathrm{Set}, R \mapsto \{(r, s) \in R \times R, f(r, s) = 0 (\in R)\}.$$

In other words, a commutative ring gets mapped to the set of solutions of the original equation, but solutions are now taken in R (as opposed to $\mathbf{Z}$). This is indeed functorial (with respect to ring homomorphisms $\varphi : R \to S$). In fact, this functor is just the corepresenting copresheaf for the ring $X := \mathbf{Z}[x, y]/f$:

$$h^X : R \mapsto h^X(R) = \mathrm{Hom}_{\mathrm{CRing}}(X, R).$$

The Yoneda lemma, applied to the category $\mathrm{CRing}^{\mathrm{op}}$ gives a fully faithful functor

$$\mathrm{CRing}^{\mathrm{op}} \subset \mathrm{Fun}(\mathrm{CRing}, \mathrm{Set}), X \mapsto h^X.$$

Thus, questions about the solutions of polynomial equations (or, similarly, systems of polynomial equations), which can be phrased in terms of morphisms in CRing, can be just as well studied in the larger category at the right hand side. The additional benefit is that the right hand side contains more objects, notably *schemes*. This construction is the first step in the *functor of points* approach to algebraic geometry.

3.2 Exercises

Exercise 3.1. Let Poly $\subset$ CRing be the full subcategory consisting of the polynomial rings $\mathbf{Z}[t_1, \dots, t_n]$. Show that the Yoneda embedding restricts to an equivalence of categories

$$\mathrm{CRing} \to \mathrm{Fun}'(\mathrm{Poly}^{\mathrm{op}}, \mathrm{Set}).$$

At the right, Fun' denotes the full subcategory of $\mathrm{Fun}(\mathrm{Poly}^{\mathrm{op}}, \mathrm{Set})$ consisting of functors preserve finite products (i.e., regarded as a contravariant functor coproducts in Poly are mapped to products in Set).

Exercise 3.2. Prove Lemma 3.5. Also show that the isomorphism is functorial in F and in X, in the sense that there is an isomorphism of functors

$$\mathrm{ev} : C^{\mathrm{op}} \times \widehat{C} \to \mathrm{Set}, (X, F) \mapsto F(X)$$

and

$$C^{\mathrm{op}} \times \widehat{C} \stackrel{y^{\mathrm{op}} \times \mathrm{id}}{\to} \widehat{C}^{\mathrm{op}} \times \widehat{C} \stackrel{\mathrm{Hom}_{\widehat{C}}}{\to} \mathrm{Set}.$$

Exercise 3.3. In the interplay of algebraic and analytic geometry, there is a functor

$$\mathrm{Sch}^{\mathrm{ft}}_{\mathbf{C}} \to \mathrm{An}$$

between the category of finite type schemes over Spec $\mathbf{C}$ and analytic spaces. See, for example, [**Neeman:Algebrai**

- Describe this functor in terms of its restriction to affine schemes.
- Describe the functor $\mathrm{AffSch}^{\mathrm{ft}}_{\mathbf{C}} = (\mathrm{CRing}^{\mathrm{ft}}_{\mathbf{C}})^{\mathrm{op}} \to \mathrm{An}$ in terms of its restriction to $\mathrm{Poly}_{\mathbf{C}}$.

Exercise 3.4. Let C be a small category and D be a category. Let $F, G \in \mathrm{Fun}(C, D)$. Show that a natural isomorphism (Definition 2.26) $\alpha : F \Rightarrow G$ is the same thing as an isomorphism $F \cong G$ in the category $\mathrm{Fun}(C, D)$.

Exercise 3.5.

- Let $P : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ be the power set functor. Describe $\mathrm{End}_{\widehat{\mathrm{Set}}}(P)$ explicitly.
- Redo Exercise 2.9 (in a single line).

Exercise 3.6. Let G be a monoid. Let $U : G - \mathrm{Set} \to \mathrm{Set}$ be the forgetful functor from the category of G-sets. Show this functor is corepresentable. Construct a bijection

$$G = \mathrm{End}_{\mathrm{Fun}(G-\mathrm{Set}, \mathrm{Set})}(U).$$

This identification is a(n easy) piece of *Tannaka duality*. Broadly speaking it answers the question how a monoid (or group) can be recovered from the category of representations.

Exercise 3.7. Let $\mathrm{GL}_n\colon \mathrm{CRing} \to \mathrm{Set}$ be the functor which sends each commutative ring R to the set of invertible $n \times n$ matrices with entries in R. Show that GL_n is corepresentable by a ring G.

A *Hopf algebra* (over $\mathbf{Z}$) is a commutative ring R together with ring homomorphisms $\Delta\colon R \to R \otimes_{\mathbf{Z}} R$ and $\varepsilon\colon R \to \mathbf{Z}$, and a $\mathbf{Z}$-linear map $\lambda\colon R \to R$, which are called, respectively, comultiplication and counit and antipode map, such that the following equalities hold: $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$, $m(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id} = m(\varepsilon \otimes \mathrm{id})\Delta$ and $m(\mathrm{id} \otimes \lambda)\Delta = \iota\varepsilon = m(\lambda \otimes \mathrm{id})\Delta$. Here $m\colon R \otimes R \to R$ and $\iota\colon \mathbf{Z} \to R$ are the multiplication and unit of R.

Show that the group structure on $\mathrm{GL}_n(\mathbf{R})$ induces a Hopf algebra structure on G. Describe the comultiplication explicitly.

Exercise 3.8. Let G be a monoid. Explain the following identification and inclusion (see Exercise 2.1 for the definition of BG):

$$G = \mathrm{Hom}_{BG}(\star, \star) = \mathrm{Hom}_{G-\mathrm{Set}}(G, G) \subset \mathrm{Hom}_{\mathrm{Set}}(G, G)$$

If G is a group, show that this implies an inclusion

$$G \subset \mathrm{Aut}_{\mathrm{Set}}(G, G)$$

of the group G in the symmetric group on G, i.e., *Cayley's theorem*.

Chapter 4

Limits and colimits

A classical lemma in topology says that a pair of continuous functions taking values in some topological space X

$$f_- : (-\infty, 0] \to X, f_+ : [0, +\infty) \to X$$

such that $f_-(0) = f_+(0)$ gives rise to a unique function

$$f : (-\infty, +\infty) \to X$$

whose restrictions are f_- and f_+. In the same vein, given say a field k and two commutative k-algebras A and B, a ring homomorphism

$$f : A \otimes_k B \to C$$

into another commutative ring C is the same as a pair of ring homomorphisms

$$a : A \to C, b : B \to C$$

whose composition with the maps $k \to A$ and $k \to B$ agrees. In both cases, we see that a map out of a given object ($(-\infty, +\infty)$, resp. $A \otimes_k B$) into any other object in the category in question consists of maps out of certain other objects that are compatible. Both notions are pushouts, which in turn is a special case of the concept of a colimit. These, and the corresponding dual notion, limits, are introduced in this chapter.

4.1 Initial and terminal objects

Definition 4.1. An object X in a category C is called an *initial object* if, for every object $Y \in C$, the Hom-set $\mathrm{Hom}_C(X, Y)$ is a singleton, i.e., has exactly one element.

A *terminal object* in C is, by definition, an initial object in C^{op}, i.e., for each $Y \in C$,

$$\mathrm{Hom}_{C^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_C(Y, X) = \{*\}.$$

It is customary, in view of the examples below, to denote initial objects by $0 \in C$ and terminal objects by $1 \in C$.

Example 4.2. We have the following examples:

category	initial object	terminal object
Set	$\varnothing$	$\{*\}$
Grp, Mon, Ab, Vect_k	0 (the trivial group, monoid, vector space)	0
Ring, CRing	$\mathbf{Z}$	0

Most of these are immediately verified. As an example, we check that the initial object of Ring is $\mathbf{Z}$. Indeed, any ring homomorphism $f : \mathbf{Z} \to R$ sends (by definition) $0 \mapsto 0_R$, $1 \mapsto 1_R$. By additivity, therefore $n \mapsto n \cdot 1_R$ for $n \geqslant 0$ and $-n \mapsto -(n \cdot 1_R)$.

Lemma 4.3. Initial and terminal objects are unique up to unique isomorphism. That is: if X and $Y \in C$ are both initial (or both terminal) objects, then there is a unique isomorphism $X \overset{\cong}{\to} Y$.

Proof. Since X and Y are initial, there are unique maps f and g as in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\mathrm{id}_X"'] & Y \arrow[d, "g"] \arrow[dr, "\mathrm{id}_Y"] & \\
& X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Since X (resp. Y) is initial both parts commute, i.e., $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. We conclude that X is uniquely isomorphic to Y. For terminal objects, apply the argument to C^{op}. □

While initial (and terminal) objects are unique (up to unique isomorphism) *if* they exist, there is no a priori reason for them to exist in a category. As a "stupid" example, the full subcategory $\mathrm{Set}_{\geqslant 2}$ of all sets that have at least 2 elements has neither an initial nor a terminal object(!) . A more meaningful counterexample is the category of fields (Exercise 4.1).

4.2 Diagrams and cones

Definition 4.4. Let C be a category and J a *small* category. A *diagram* of type J (or of shape J) in C is a functor

$$D : J \to C.$$

For somewhat historic reasons, one tends to denote the objects of the indexing category J by i, j and write D_i instead of $D(i)$ etc.

Example 4.5. Here are some basic examples. In the source categories, the only non-identity morphisms are for the indicated inequalities.

J	diagrams of type J
$\{0\}$	objects in J
$\{0 < 1\}$	$D_0 \to D_1$, i.e., morphisms in C
$\{0 < 1 < 2\}$	$D_0 \to D_1 \to D_2$, i.e., composable morphisms in C
$\{0\} \sqcup \{1\}$	(D_0, D_1), i.e., pairs of objects in C
$\{0 \rightrightarrows 1\}$	$D_0 \underset{f}{\overset{g}{\rightrightarrows}} D_1$, i.e., two objects with two parallel morphisms between them

Definition 4.6. Given a category J, the *left cone* $J^{\lhd}$ is defined as the category whose objects are

$$\mathrm{Obj}(J^{\lhd}) = \{*\} \sqcup \mathrm{Obj}(J),$$

i.e., we add a new object. Moreover, this object is turned into an initial object:

$$\mathrm{Hom}_{J^{\lhd}}(*, *) = \mathrm{Hom}_{J^{\lhd}}(*, j) := \{*\}$$

for each $j \in J$. The morphisms in J itself are untouched, i.e., $\mathrm{Hom}_{J^{\lhd}}(i, j) := \mathrm{Hom}_J(i, j)$ for $i, j \in J$. Finally, $\mathrm{Hom}_{J^{\lhd}}(j, *) := \varnothing$. The identity and composition laws are the obvious ones, noting that we have no choice how to compose morphisms out of $*$ (there is only one such morphism), nor into $*$ (there are no such morphisms).

Definition 4.7. Let $D : J \to C$ be a diagram (if we speak of diagrams by definition this includes the assumption that J is a *small* category). The category of *cones* of D, denoted $\mathrm{Cone}(D)$ is defined as the (non-full) subcategory of the category of functors

$$\mathrm{Fun}(J^{\lhd}, C)$$

consisting of those objects (i.e., functors), whose restriction to $J \subset J^{\lhd}$ is the given diagram D and of those morphisms (i.e., natural transformations), whose restriction to J is the identity of D.

Thus, the additional information present in a cone $\tilde{D} : J^{\lhd} \to C$ is an extra object $X \in C$, namely $X = \tilde{D}(*)$. For each $j \in J$, this object comes with maps $f_j : X \to D_j$ such that for each morphism $\alpha : i \to j$ in J, the diagram

$$\begin{tikzcd} & X \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ D_i \arrow[rr, "D_\alpha"] & & D_j \end{tikzcd}$$

commutes. A morphism between such a cone, and another similar one (with X', f'_i etc.) is a map $X \to X'$ such that the diagrams

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f_i"] & & X' \arrow[dl, "f'_i"] \\ & D_i & \end{tikzcd}$$

commute for all $i \in J$.

Example 4.8. We further demystify this with some concrete examples:

J	$D : J \to C$	$\mathrm{Obj}(\mathrm{Cone}(D))$
$\{0\}$	$D_0 \in C$	$X \to D_0$ (i.e., an object X and a morphism to the given D_0)
$\{0 < 1\}$	$D_0 \to D_1$	$\begin{tikzcd} & X \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\ D_0 \arrow[rr] & & D_1 \end{tikzcd}$, i.e., an object X, two maps f_0, f_1 such that the diagram commutes
$\{0\} \sqcup \{1\}$	(D_0, D_1)	$X \in C$, $f_0 : X \to D_0$, $f_1 : X \to D_1$, i.e., an object X, two maps f_0, f_1 with no condition on them.
$\{0 \rightrightarrows 1\}$	$D_0 \overset{g}{\underset{f}{\rightrightarrows}} D_1$	$X \xrightarrow{a} D_0 \overset{g}{\underset{f}{\rightrightarrows}} D_1$, i.e., an object X, a morphism a such that $f \circ a = g \circ a$.

The morphisms in $\mathrm{Cone}(D)$ are the expected ones. For example, for $D_0 \overset{g}{\underset{f}{\rightrightarrows}} D_1$ a morphism between two cones

$$\mathrm{Hom}_{\mathrm{Cone}(D)}(X \xrightarrow{a} D_0 \overset{g}{\underset{f}{\rightrightarrows}} D_1, Y \xrightarrow{b} D_0 \overset{g}{\underset{f}{\rightrightarrows}} D_1)$$

is a map $x : X \to Y$ such that the diagram involving x, a and b commutes.

4.3 Limits

Definition 4.9. Let $D : J \to C$ be a functor. A cone over D, i.e., an object in $\mathrm{Cone}(D)$ is called a *limit* of D if it is a terminal object of $\mathrm{Cone}(D)$.

As a special case of Lemma 4.3 we have:

Corollary 4.10. Let $D : J \to C$ be given. If a limit of D exists, it is unique up to unique isomorphism.

Remark 4.11. In view of the unicity one also speaks of *the* limit and denotes it by $\lim D$. It is also customary to indicate the functor D by drawing its image in C. For example,

$$\lim(D_0 \overset{g}{\underset{f}{\rightrightarrows}} D_1) := \lim(D : J \to C),$$

where D is the functor $\{0 \rightrightarrows 1\} \to C$ mapping to the previously displayed objects and morphisms in C.

Example 4.12. We revisit the examples above:

For $J = \{0\} \xrightarrow{D} C$, a limit is a terminal object in the category whose objects are maps $X \to D_0$ and whose morphisms are the obvious commutative triangles. This category contains, in particular, the identity morphism $D_0 \xrightarrow{\mathrm{id}_{D_0}} D_0$. This object is terminal: given any $X \xrightarrow{f_0} D_0$, there is a unique dotted map making the triangle commute:

$$\begin{tikzcd} X \arrow[rr, dotted, "f_0"] \arrow[dr, "f_0"'] & & D_0 \arrow[dl, equal, "\mathrm{id}_{D_0}"] \\ & D_0 & \end{tikzcd}.$$

Thus a limit for J is just the given object D_0 (together with the only possible map as part of the cone, namely the identity). We observe that 0 is the initial object of J, so this somewhat boring result is a special case of a more general phenomenon considered in Exercise 4.3.

This exercise shows that the limit of a diagram $D_0 \to D_1$ (of shape $0 < 1$) in any category is just the object D_0 together with the obvious maps, namely the identity id_{D_0} and the given map to D_1.

Definition 4.13. A limit of a diagram $D : J = \{0\} \sqcup \{1\} \to C$ is called a *binary product* or just *product*. If it exists, it is also denoted as $D_0 \times D_1$ (given that such a diagram is nothing but a pair of objects D_0 and $D_1 \in C$).

Concretely, a product of such a diagram is an object X equipped with two maps $X \to D_0$, $X \to D_1$ such that for any other object Y coming with two maps $Y \to D_0$, $Y \to D_1$ there is a unique map $Y \to X$ making the obvious diagrams commute. To say more about this, we need to look at some specific categories:

Example 4.14. For $C = \mathrm{Set}$, the product is just the usual product of sets: $D_0 \times D_1 = \{(d_0, d_1) | d_i \in D_i\}$: it comes with two projections $p_i : D_0 \times D_1 \to D_i$, and given any other set Y with two maps $f_i : Y \to D_i$, there is a unique map

$$f : Y \to D_0 \times D_1, y \mapsto (f_0(y), f_1(y))$$

such that $p_i \circ f = f_i$.

For $C = \mathrm{Top}$, the product $D_0 \times D_1$ is the set-theoretical product, equipped with the product topology. Indeed, this topology on $D_0 \times D_1$ is such that the two projections $D_0 \times D_1 \to D_i$ are continuous, and such that for any two continuous maps $f_i : Y \to D_i$ (from any topological space Y), the resulting map f as above is again continuous. The (easy) proof of this fact is in any topology textbook.

Example 4.15. In the same vein, for $C = \mathrm{Vect}_k$, the product is the usual product of vector spaces. We will understand the observation that the product in these two categories is "just" the product in Set equipped with some extra structure more fully once we see that the forgetful functor $\mathrm{Vect}_k \to \mathrm{Set}$ does not come alone, but has a so-called left adjoint (see Example 5.3) which automatically forces the forgetful functor to preserve products (see Lemma 5.7).

Example 4.16. In the category of fields Fields($\subset$ CRing), there is no product $K \times L$ if the two fields have different characteristic, since there are no maps between fields of different characteristic. One can also show (easily) that there is no product even of the form $K \times K$ if K admits a non-identity automorphism.

By contrast, the category Ring (or also CRing) does have products, given by the set-theoretic product together with the component-wise addition and multiplication.

Product of vector spaces exist for more than two factors. They are special cases of the following:

Definition 4.17. Let I be a set, regarded as a category in which we only have identity morphisms (this is called a *discrete category*). A limit of a functor $D : I \to C$ is called a *product*. It is also denoted as $\prod_{i \in I} D_i$, using that such a functor D is just a collection of objects $D_i \in C$ (for each $i \in I$).

Note that an empty product (the case $I = \varnothing$) is just an initial object. All we said so far about Set, Mon, Mod_R, Top, CRing, Ring (existence and description of binary products) extends to arbitrary products (indexed by sets) without any change.

It is useful to picture these notions: a product is an object $\prod_i D_i$ with maps p_i such that for any other object T mapping to all the D_i there is a unique dotted map making everything commutative, like so:

$$\begin{tikzcd}
& \prod_i D_i \arrow[dl, "p_i"'] \arrow[d, "p_j"] \arrow[dr] \arrow[drr] & & T \arrow[lll, dotted, "\exists!"'] \arrow[dllll] \arrow[dll] \arrow[dl] \arrow[d] \\
D_i & D_j & D_k & \dots
\end{tikzcd}$$

Definition 4.18. A limit of a diagram $\{0 \rightrightarrows 1\} \to C$ is called an *equalizer*.

Example 4.19. Again, in a general category, equalizers need not exist. They do exist in Set, where an equalizer of a diagram $D_0 \overset{g}{\underset{f}{\rightrightarrows}} D_1$ is given by(!)

$$\{d \in D_0, f(d_0) = g(d_0)\}.$$

Equalizers also exist in Mon, Grp, Mod_R (modules over a ring R), Top (where they are in each case the set-theoretic product equipped with the usual monoid structure, ... , topology).

The following little lemma will be used later in the construction of adjoint functors (Theorem 5.19).

Lemma 4.20. Suppose we have an equalizer

$$E = \mathrm{eq}(X \overset{f}{\underset{g}{\rightrightarrows}} Y)$$

in a category (i.e., assume the equalizer exists). Then the natural map $E \to X$ is a monomorphism.

Proof. We have
$$\mathrm{Hom}(T, E) = \{t \in \mathrm{Hom}(T, X), ft = gt\} \subset \mathrm{Hom}(T, X).$$

Definition 4.21. A limit of shape

$$\begin{tikzcd}
& \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

is called a *fiber product*. A commutative diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}$$

is called *cartesian* if the natural map $T \to A \times_C B := \lim(A \to C \leftarrow B)$ is an isomorphism.

Example 4.22. In Set, the fiber product of a diagram $A \xrightarrow{f} C \xleftarrow{g} B$ is the set $\{(a, b) \in A \times B, f(a) = g(b)\}$.

If a category C has fiber products, then a map $f : X \to Y$ is a monomorphism iff the natural map Δ is an isomorphism. This follows directly(!) from the definition of monomorphisms and the universal property of the fiber product.

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Equivalently, f is a monomorphism iff the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is cartesian.

4.4 Colimits

Definition 4.23. A *colimit* of a diagram $D : J \to C$ is, by definition, a limit of the resulting diagram

$$D^{\mathrm{op}} : J^{\mathrm{op}} \to C^{\mathrm{op}}.$$

A *coproduct* is a colimit of a diagram $D : J \to C$ where J is a discrete category (has only identity morphisms). It is then denoted $\coprod_{j \in J} D_j$.

A *coequalizer* is a colimit of a diagram $D : \{0 \rightrightarrows 1\} \to C$.

A *pushout* is the dual notion for a pullback, i.e., a colimit of a diagram of shape

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet. & \end{tikzcd}$$

Thus, a colimit is just a special case of a limit (and vice versa). However, since for many specific categories C, one does not have a handy description of C^{op}, it is in practice quite useful to understand this concept independently from limits, so we now unwind these notions.

Let J be a discrete small category and $D : J \to C$ a functor. I.e., just a collection of objects D_j, for each $j \in J$ (which is a set). A coproduct $\coprod_j D_j$ is then a product in the opposite category C^{op}. Thus, it is an object $\coprod_i D_i$ with maps p_i such that for any other object T mapping from all the D_i there is a unique dotted map making everything commutative, like so:

$$\begin{tikzcd} & \coprod_i D_i \arrow[rr, dotted, "\exists!"] & & T \\ D_i \arrow[ur, "p_i"] \arrow[urrr] & D_j \arrow[u, "p_j"] \arrow[urr] & D_k \arrow[ul] \arrow[ur] & \dots \arrow[ull] \arrow[u] \end{tikzcd}$$

Again, coproducts and other colimits need not exist. If a colimit for a given diagram $D : J \to C$ does exist, it is unique up to unique isomorphism.

Example 4.24. In Set, arbitrary (small) coproducts exist. They are given by the disjoiunt union (whence the notation $\coprod_i$). The disjoint union of topological spaces $\coprod X_i$ (in which U is open iff $U \cap X_i$ is open in X_i for each i) is the coproduct of topological spaces.

In Mon and Grp, coproducts exist as well. The adjoint functor theorem (see Theorem 5.19 and its corollaries) will construct coproducts in Grp for free. That said, let us only indicate that the coproduct in Grp is given by the so-called *free product* (which really is *not* a product): given two groups G, H the elements in $G \sqcup H$ are words with letters from both groups except the identity elements, modulo the relations $xx^{-1} = \varnothing$ for any two adjacent copies of $x \in G \cup H$ and its inverse.

In Ab and Vect_k, and more generally Mod_R (modules over a ring R), coproducts are given by direct sums

$$\coprod_{i \in I} V_i = \{(v_i) | v_i \in V_i, \text{only finitely many } v_i \neq 0\}.$$

In CRing, coproducts exist as well. The coproduct $R \sqcup S$ is commonly known as the *tensor product* $R \otimes_{\mathbf{Z}} S$. Indeed, by definition of the tensor product $\mathrm{Hom}(R \otimes S, T) = \mathrm{Hom}(R, T) \times \mathrm{Hom}(S, T)$ for any commutative ring. More generally, the coproduct in CRing_k (commutative k-algebras) is given by $- \otimes_k -$.

Infinite coproducts exist as well, and are referred to as infinite tensor products in the literature (see e.g., [**Bourbaki:Algebra**]). See also Exercise 4.7.

4.5 Functoriality

In the discussion of (co)limits, we have so far fixed a given diagram

$$D : J \to C.$$

We will now lift this restriction and discuss how (co)limits behave when D varies. *Therefore, in this section, we fix a small category J and a (not necessarily small) category C.*

Definition 4.25. The *diagonal functor*

$$\Delta : C \to \mathrm{Fun}(J, C)$$

sends an object $X \in C$ to the functor $(J \ni) j \mapsto X$ and any morphism $j \to j'$ in J is mapped to id_X.

Given $D \in \mathrm{Fun}(J, C)$, a cone on D is then nothing else but an object $X \in C$ and a map of diagrams

$$\Delta(X) \to D.$$

Lemma 4.26. Suppose that C has all J-shaped limits. Then the choice of a limit $\lim D$ for each diagram $D : J \to C$ assembles to a functor

$$\lim : \mathrm{Fun}(J, C) \to C.$$

Proof. In order to specify this functor, we need to define it on objects, which we already did, and on morphisms. Let us be given a morphism between diagrams, i.e., a natural transformation $\alpha : D \to D'$ (of functors $J \to C$). (We write $D \to D'$ as opposed to $D \Rightarrow D'$ just to streamline the notation.) The limits $\lim D$ and $\lim D'$ give rise to the vertical maps in the following diagram

$$\begin{tikzcd} \Delta(\lim D) \arrow[r, dotted] \arrow[d, "\iota"] & \Delta(\lim D') \arrow[d, "\iota'"] \\ D \arrow[r, "\alpha"'] & D'. \end{tikzcd}$$

The composite $\alpha \circ \iota$ is a cone on the diagram D'. As $\lim D'$ is a terminal cone (being a limit), there is a unique map of diagrams as depicted making the diagram commutative. We define this map to be the image (under lim) of our map α.

By the unicity of such maps, it is clear that lim preserves composition and identity, so it indeed gives a functor. □

Example 4.27. Consider the category $J = \{1 \to 0 \leftarrow 2\}$. Recall that a limit of a diagram $D : J \to \mathrm{Top}$ is the pullback of a pair of continuous maps

$$\begin{tikzcd} X_1 \times_{X_0} X_2 \arrow[r] \arrow[d] & X_1 \arrow[d, "f_1"] \\ X_2 \arrow[r, "f_2"] & X_0 \end{tikzcd}$$

(a choice of a pullback is the subspace of the product $\{(x_1, x_2) \in X_1 \times X_2, f_1(x_1) = f_2(x_2)\}$, equipped with the subspace topology of the product topology). In this case the lemma says that given a collection of continous maps $X_i \to X'_i$ $(i = 0, 1, 2)$ where $X'_1 \to X'_0 \leftarrow X'_2$ is another similar diagram such that the obvious squares commute), there is a unique continuous map

$$X_1 \times_{X_0} X_2 \to X'_1 \times_{X'_0} X'_2$$

that is compatible with the given maps $X_i \to X'_i$.

4.6 Preservation of limits

Definition 4.28. Let J be a small category. We say that a functor $F : C \to D$ *preserves limits of shape* J if for any diagram $K : J \to C$ and any limit cone over K, the image of this cone is a limit cone for the composition $F \circ K : J \to D$. We write this slightly colloquially as

$$F(\lim K_j) = \lim F(K_j).$$

We say F *preserves limits* if this holds for all small categories J.

Recall that any two limit cones of K are (uniquely) isomorphic (both being terminal objects in Cone K, Lemma 4 Since any functor preserves isomorphisms (Lemma 2.20), it is therefore enough to check the above condition for *some* (as opposed to all) limit cone over K.

Example 4.29. The forgetful functor $\mathrm{Vect}_k \to \mathrm{Set}$ preserves limits. By Proposition 4.32 below it suffices to check this for equalizers and products. Indeed, the equalizer of a diagram

$$V \underset{g}{\overset{f}{\rightrightarrows}} W$$

is the subspace $\{v \in V | f(v) = g(v)\}$ (with the obvious inclusion into V), and this is also the equalizer of the underlying diagram of sets. Similarly, the underlying set of a product of vector spaces is the product of the underlying sets.

Lemma 4.30. Let C be a category and $X \in C$ arbitrary. Then the corepresenting functor

$$h^X : C \to \mathrm{Set}, Y \mapsto \mathrm{Hom}_C(X, Y)$$

preserves limits. Written colloquially:

$$\mathrm{Hom}_C(X, \lim Y_i) = \lim \mathrm{Hom}_C(X, Y_i).$$

Likewise, the representing functor

$$h_X : C^{\mathrm{op}} \to \mathrm{Set}$$

preserves limits. Again, written colloquially (and using that limits in C^{op} are colimits in C):

$$\mathrm{Hom}_C(\operatorname{colim} Y_i, X) = \lim \mathrm{Hom}_C(Y_i, X).$$

Proof. Let $D : I \to C$ be a diagram whose limit $\lim D$ exists. A morphism $X \to \lim D$ is a collection of compatible morphisms $f_i : X \to D(i)$ for each $i \in I$ that are compatible. This is the same as an element in $\lim \mathrm{Hom}_C(X, D_i)$.

The dual statement follows since the representing functor $h_{X,C} : C^{\mathrm{op}} \to \mathrm{Set}$ is the same as the corepresenting functor $h^X_{C^{\mathrm{op}}} : C^{\mathrm{op}} \to \mathrm{Set}$ for X, regarded as an object in the opposite category. □

Corollary 4.31. Representable functors $C^{\mathrm{op}} \to \mathrm{Set}$ (Definition 2.29) and corepresentable functors $C \to \mathrm{Set}$ preserve limits.

We will establish a converse statement for this in the context of the adjoint functor theorem.

4.7 Basic existence results

The following criterion constructs arbitrary limits from special ones. In particular it is a means to show the existence of arbitrary limits.

Proposition 4.32. Let $D : J \to C$ be a diagram. Consider the diagram

$$\prod_{k \in J} D_k \underset{g}{\overset{f}{\rightrightarrows}} \prod_{\alpha : i \to j} D_j,$$

where the component corresponding to a morphism $\alpha : i \to j$ is

- $p_j : \prod_{k\in J} D_k \to D_j$ for the first map, and
- $D_\alpha \circ p_j : \prod_{k\in J} D_k \to D_i \xrightarrow{D_\alpha} D_j$ for the second map.

We suppose that all these products exist. If, moreover, the equalizer E of the above diagram exists, then this equalizer is a limit of D.

Proof. For $X \in C$, $\mathrm{Hom}_C(X, E)$ identifies with morphisms $h : X \to \prod_k D_k$ such that $f \circ h = g \circ h$. Moreover, h is tantamount to a familiy of morphisms $h_k : X \to D_k$. The equality $fh = gh$ is then equivalent to $(fh)_\alpha = (gh)_\alpha$ for each $\alpha : i \to j$, so that $p_j h = D(\alpha)p_i h$ and then $h_j = D_\alpha h_i$. All in all, this is a familiy of morphisms $h_k : X \to D_k$ such that $h_j = D_\alpha h_i$ for all α. This is precisely the datum of a cone over D. This shows that E is a limit for D. □

Definition 4.33. A category is called *complete* (resp. *cocomplete*) if *all* diagrams $D : J \to C$ (with arbitrary small categories J) admit a limit (resp. colimit).

The proposition thus shows that a category is (co-)complete if (and obviously also only if) it has (co-)equalizers and (co-)products. See also Exercise 4.7 for a statement further reducing (co-)products to finite (co-)products and cofiltered limits (resp. filtered colimits).

The categories Set, Mon, Grp, Ab, Mod_R, Top, Ring and CRing are all complete and cocomplete. The fact that both completeness and cocompleteness hold parallely is not a coincidence: we will show – as a consequence of the adjoint functor theorem) – that, under a mild additional set-theoretic condition (known as accessibility) a category is complete iff it is cocomplete.

The existence of (co)limits is passed on to the category of functors as follows:

Lemma 4.34. Let J, C be small categories and D be a category. Suppose that D has all limits (resp. colimits) of shape J. Then the functor category $\mathrm{Fun}(C, D)$ also has all limits (resp. colimits) of shape J.

Moreover, these (co)limits are computed objectwise, i.e., for each $X \in C$ the evaluation-at-X-functor

$$\mathrm{ev}_X : \mathrm{Fun}(C, D) \to D, F \mapsto F(X)$$

preserves (co)limits. Written more colloquially, the following natural maps are isomorphisms:

$$(\lim F_j)(X) \to \lim(F_j(X)), \operatorname{colim}(F_j(X)) \to (\operatorname{colim} F_j)(X).$$

Proof. It suffices to do limits, since colimits are limits in D^{op} and $\mathrm{Fun}(C, D^{\mathrm{op}}) = \mathrm{Fun}(C^{\mathrm{op}}, D)$.

We colloquially write F_j ($j \in J$) for the datum of a functor $F : J \to \mathrm{Fun}(C, D)$. In order to construct a limit for F, we need to specify an object E in $\mathrm{Fun}(C, D)$. For each object $X \in C$, the composite

$$F_\bullet(X) : J \to \mathrm{Fun}(C, D) \xrightarrow{\mathrm{ev}_X} D, j \mapsto F_j \mapsto F_j(X),$$

(i.e., ev_X is the evaluation at the object X). has a limit by assumption on D. Choose such a limit and define $E(X)(\in D)$ to be that chosen limit. The limit is functorial in X by Lemma 4.26, so any map $X \to X'$ in C gives rise to a unique map $E(X) \to E(X')$ that is compatible with the given maps $F_j(X) \to F_j(X')$ for all $j \in J$. (More formally, a morphism $X \to X'$ yields a natural transformation $F_\bullet(X) \to F_\bullet(X')$ of diagrams $J \to D$. Then apply Lemma 4.26). By unicity, E is then necessarily a functor.

One checks(!) by unwinding the definitions that E is indeed a limit. □ (!)

Corollary 4.35. For a small category C, the category $\widehat{C} := \mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ has all (co)limits.

According to Exercise 4.16, the Yoneda embedding

$$C \to \widehat{C}$$

preserves all limits, but does not usually preserve colimits.

4.8 Filtered colimits and compact objects

Filtered categories and the concomitant filtered colimits and cofiltered limits are pervasive throughout mathematics. They are the basis of finiteness conditions on objects in an abstract category. For example, linear algebra has the condition that an R-module is finitely presented, field theory the condition that a field extension is finitely generated, or topology has the condition that a topological space is compact. While many expositions of these concepts spell these conditions out "explicitly", we will see that they are in fact special cases of the same category-theoretic notion.

Definition 4.36. A category C is *finite* if it has finitely many objects and morphisms.

Definition and Lemma 4.37. A category C is called *filtered* if every finite diagram $D : J \to C$ (i.e., J is finite) has a cocone. This is equivalent to the following three conditions (these amount to the cases $J = \varnothing$, $J = \{0, 1\}$ and $J = \{0 \rightrightarrows 1\}$).

- $C \neq \varnothing$,
- for any two objects $X, Y \in C$, there is an object $Z \in C$ together with morphisms $X \to Z \leftarrow Y$.
- for any two morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in C, there is an object Z and a morphism $h : Y \to Z$ such that $hf = hg$:

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{h} Z.$$

Remark 4.38. • Note the datum of h in the last condition resembles the situation with a coequalizer. However, the condition here is weaker: the cocone is not required to be initial.

- If C happens to be (the category associated to) a poset, then the third condition is vacuous since necessarily $f = g$. A (category associated to a) poset P is also called *directed* if it is filtered. Equivalently $P \neq \varnothing$ and for any $x, y \in P$ there needs to be a $z \in P$ with $x \leqslant z$, $y \leqslant z$.

Definition 4.39. A diagram $D : I \to C$ in a category C is a *filtered diagram* (resp. *directed diagram*) if I is filtered (resp. if I is a directed poset). A *filtered colimit* is the colimit of a filtered diagram $D : I \to C$. To emphasize the fact that I is directed, many authors denote directed colimits by

$$\varinjlim D \quad \text{or also as} \quad \varinjlim X_i$$

and refer to such colimits as *inductive limits*. We will avoid this terminology to avoid confusion of such special colimits with limits.

As usual, the condition on the opposite categories or diagrams is denoted with a "co": a *cofiltered diagram* $D : I \to C$ is one where I is a *cofiltered category*, i.e., I^{op} is filtered etc.

A codirected limit is, in older literature, also known as a *projective limit*, and denoted

$$\varprojlim D \quad \text{or also as} \quad \varprojlim X_i$$

Example 4.40. • The natural numbers $\mathbf{N}$ are a directed poset. Diagrams $\mathbf{N} \to C$ can be depicted as

$$X_0 \to X_1 \to X_2 \to \dots.$$

This is an example of a directed diagram.

Dually, diagrams $\mathbf{N}^{\mathrm{op}} \to C$ can be depicted as

$$\dots X_2 \to X_1 \to X_0.$$

This is a codirected diagram.

- More generally, any preordered set $(A, \leqslant)$ in which any finite subset (including the empty set) has an upper bound, gives a directed poset.

- The category Idem is the category with one object $\star$, and one non-identity morphism $e : \star \to \star$ satisfying
$$e \circ e = e.$$
This category is an example of a filtered, but non-directed category.

Lemma 4.41. Let $D : J \to \mathrm{Set}$ be a filtered diagram. Then the colimit of D exists, and it is given by
$$\coprod_{i \in J} D_i / \sim,$$
where $\sim$ is the equivalence relation $x_i (\in D_i) \sim x_j (\in D_j)$ if and only if there is some $k \in J$ and maps $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$ such that $D(\alpha)(x_i) = D(\beta)(x_j) (\in D_k)$.

Informally, two elements in the disjoint union of the sets D_i are identified if and only if they agree "eventually" in some D_k for "large enough" k.

Proof. We first check that $\sim$ is an equivalence relation. It is clearly reflexive and symmetric. For transitivity, let $x_1 \in D_{i_1}$, $x_2 \in D_{i_2}$, $x_3 \in D_{i_3}$ such that $x_1 \sim x_2$, $x_2 \sim x_3$. Thus there are objects i_4 and i_5 and maps as indicated in the diagram

$$\begin{tikzcd}
i_1 \arrow[dr] & & i_2 \arrow[dl] \arrow[dr] \arrow[dd, dotted] & & i_3 \arrow[dl] \\
& i_{12} \arrow[dr, dotted] & & i_{23} \arrow[dl, dotted] & \\
& & i, & &
\end{tikzcd}
\qquad
\begin{tikzcd}
x_1 \arrow[dr, mapsto] & & x_2 \arrow[dl, mapsto] \arrow[dr, mapsto] \arrow[dd, mapsto] & & x_3 \arrow[dl, mapsto] \\
& x_{12} \arrow[dr, mapsto] & & x_{23} \arrow[dl, mapsto] & \\
& & x. & &
\end{tikzcd}$$

As J is filtered, the finite diagram consisting of $i_1, i_2, i_3, i_{12}, i_{23}$ has a cocone, denoted i. The image of x_{12} and x_{23} in D_i agree since they both are the image of x_2. This shows $x_1 \sim x_3$.

Let $C := \coprod D_i / \sim$ as in the claim. We have natural maps $s_i : D_i \to C$ which we denote by $x \mapsto [x]$. These clearly exhibit C as a cocone. Suppose given another cocone $(t_i : D_i \to M)$ on our diagram D. Clearly there is at most one map $t : C \to M$ such that $t \circ s_i = t_i$, since the D_i map (jointly) surjective to C. To check the existence of such a map, define
$$t : C \to M, t([x]) := t_i(x).$$
One checks immediately this is well-defined by definition of $\sim$ and satisfies $t \circ s_i = t_i$. $\square$

Recall from Lemma 4.30 that corepresentable functors $h^X = \mathrm{Hom}_C(X, -)$ preserve limits (in C). Whether or not corepresentable functors preserve *colimits* depends very much on the category C, on the object X, and on the type of colimits allowed. We now discuss the most important class of colimits that corepresentable functors *may* preserve. A variant of this appears in Exercise 4.14.

Definition 4.42. Let C be a category that has all filtered colimits. An object $X \in C$ is called *compact* or *finitely presentable* if the functor $h^X : \mathrm{Hom}_C(X, -)$ preserves filtered colimits. I.e., for any filtered diagram $D : I \to C$, the natural map
$$\operatorname{colim} \mathrm{Hom}_C(X, D_i) \to \mathrm{Hom}_C(X, \operatorname{colim} D_i)$$
is a bijection.

The colimit at the left is a colimit in Set, whereas the one in the right is in C. The surjectivity of the map means that given a map (in C)
$$f : X \to \operatorname{colim} D_i$$

there is some $j \in J$ and a map f' such that the following diagram commutes:

$$\begin{tikzcd} & D_j \arrow[d] \\ X \arrow[ur, dotted, "f'"] \arrow[r, "f"] & \operatorname{colim} D_i. \end{tikzcd}$$

Thus X is "small enough" so that any map out of X factors over a finite piece of the colimit. The following examples corroborate the idea that compactness is really a finiteness condition.

Example 4.43.

- A set $X \in \mathrm{Set}$ is compact iff its cardinality is finite. First of all, a singleton $\{*\}$ is compact since $\mathrm{Hom}(\{*\}, D_i) = D_i$. More generally, the compactness of any finite set $X = \coprod_{\text{finite}} \{*\}$ then follows either from direct inspection(!) or, alternatively from Corollary 4.47 below.

  Conversely, let X be a compact object in Set. We observe

  $$X = \bigcup_{U \subset X \text{ finite}} U.$$

  This union is in fact a filtered colimit (even directed): for any *finite* family of finite subsets $U_i \subset X$ (i runs through a finite index set), the union $\bigcup_i U_i$ is again a finite subset of X). Using the bijection

  $$\operatorname{colim} \mathrm{Hom}_{\mathrm{Set}}(X, U) = \mathrm{Hom}_{\mathrm{Set}}(X, \operatorname{colim} U) = \mathrm{Hom}_{\mathrm{Set}}(X, X) \ni \mathrm{id}_X,$$

  and the description of filtered colimits in Lemma 4.41, we see that there is some finite subset $U \subset X$ and a map $f : X \to U$ such that its composition with the inclusion $U \subset X$ equals id_X. This forces $U = X$ and in particular that X is finite.

- Let X be a topological space. Then X is compact in the sense of topology (i.e., every open covering of X has a finite subcover) iff X is compact in the poset $\mathrm{Open}(X)$ consisting of the open subsets of X, ordered by inclusion.

  To see this, note first that to a set $U_i, i \in I$ of open subsets we can assign a filtered diagram

  $$K : \{\text{finite subsets of } I\} \to \mathrm{Open}(X),$$

  which sends any finite subset $J \subset I$ to $\bigcup_{i \in J} U_i$. This diagram is filtered and its colimit is $\bigcup_{i \in I} U_i$(!) .

  Now if $X \in \mathrm{Open}(X)$ is a compact object, and $X = \bigcup_{i \in I} U_i$ is an open covering, then id_X is a map $X \to \operatorname{colim} K$, where K is as above. By compactness it factors over some $K(J)$, i.e., over $\bigcup_{i \in J} U_i$ for a finite J. Thus X is compact in the sense of topology.

  Conversely, assume X is compact in the sense of topology and let $U : I \to \mathrm{Open}(X), i \mapsto U_i$ be any filtered diagram. We have to show

  $$\mathrm{Hom}_{\mathrm{Open}(X)}(X, \operatorname{colim} U) = \operatorname{colim}_i \mathrm{Hom}_{\mathrm{Open}(X)}(X, U_i).$$

  The left hand side is empty iff $\operatorname{colim} U = \bigcup_{i \in I} U_i \subsetneq X$, otherwise it is a singleton. In the former case, all the $U_i \subsetneq X$, so that the Hom-sets in the right hand colimit are all empty, hence so is their colimit. In the latter case there is, by compactness of X, a finite subset $J \subset \mathrm{Obj}(I)$ such that $X = \bigcup_{i \in J} U_i$. Since the diagram U is filtered, there is some cocone c of J (in the filtered category I). Then $U_c \supset \bigcup_{i \in J} U_i = X$, and we are done.

- As a bit of a terminological mismatch, let us point out that an object $X \in \mathrm{Top}$ is compact (as in Definition 4.42) iff it is a discrete topological space associated to a finite set (Exercise 4.11).

- The one-dimensional Banach space $\mathbf{C} \in \mathrm{Ban}_{\leqslant 1}$ is *not* compact. Using $l_1 = \coprod_{i \in \mathbf{N}} \mathbf{C} = \operatorname{colim}(\coprod_{i=1}^{n} \mathbf{C})$ gives a counter-example to the compactness condition. First observe

  $$\mathrm{Hom}_{\mathrm{Ban}_{\leqslant 1}}(\mathbf{C}, V) = B_1(V).$$

Then note that

$$\operatorname{colim} B_1(\mathbf{C}^n) = \{(x_n), x_n \in \mathbf{C}, \sum |x_n| \leqslant 1, \text{almost all } x_n = 0\}$$

is strictly contained in $B_1(l_1) = \{(x_n), x_n \in \mathbf{C}, \sum |x_n| \leqslant 1\}$.

A relaxed compactness condition, known as ω_1-compactness, does hold though. We will come back to this in the discussion of the adjoint functor theorem for locally presentable categories.

4.9 Commuting (co)limits with (co)limits

In analysis, there is *Fubini's theorem* stating that (under appropriate assumptions on the function) for a function

$$f : \mathbf{R} \times \mathbf{R} \to \mathbf{R},$$

the integral satisfies

$$\int_{\mathbf{R}\times\mathbf{R}} f(x,y)d(x,y) = \int_{\mathbf{R}} \left(\int_{\mathbf{R}} f(x,y)dx \right) dy.$$

In category theory, Fubini's theorem takes the following formulation:

Lemma 4.44. Let I and J be small categories and $F : I \times J \to C$ be a diagram. We write $\lim_{j\in J} F(i,j)$ for the limit of the composite

$$J \stackrel{i\times\mathrm{id}}{\to} I \times J \stackrel{F}{\to} C$$

etc. Suppose that all limits in the following expression exist:

$$\lim_i \lim_j F(i,j)$$

Then this is a limit for F.

In particular, if also all limits in the expression $\lim_j \lim_i F(i,j)$ exist, then this is isomorphic to $\lim_i \lim_j F(i,j)$, i.e., one can swap the order of performing limits in two variables.

Dually, the same statement holds with colimits instead of limits throughout.

Proof. This can be shown by reducing to the case $C = \mathrm{Set}$ via the Yoneda lemma (Lemma 3.6) and direct inspection in that case, such as done in [Rie17, Theorem 3.8.1].

We will be able to prove this more conveniently after introduce a little more terminology by the following steps, which are relevant in their own right:

- the limit functor is a partial right adjoint to the diagonal functor $\Delta : C \to \mathrm{Fun}(I,C)$ (Lemma 5.10),
- trivially, the diagram involving the diagonal functors

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\Delta"] & \mathrm{Fun}(I,C) \arrow[d, "\Delta"] \\ \mathrm{Fun}(J,C) \arrow[r, "\Delta"] & \mathrm{Fun}(I \times J, C) \end{tikzcd}$$

  commute,
- therefore, their (partial) right adjoints commute up to isomorphism (Lemma 5.11). □

While the above proof is not a one-line-proof, it is still "trivial" in the sense that it does not use any assumptions on I, J and F (other than the statement to make sense in the first place). A sharper question is the compatibility of *colimits* with *limits* (or, dually, limits with colimits). Let us be given a diagram

$$F : P \times J \to C.$$

Assuming that all limits and colimits in the following line exist, we have a canonical map

$$\kappa : \operatorname{colim}_j \lim_p F(p,j) \to \lim_p \operatorname{colim}_j F(p,j).$$

It is the unique map fitting into the diagram

$$\begin{tikzcd} F(p,j) \arrow[d] & \lim_p F(p,j) \arrow[l] \arrow[r] \arrow[d, dotted, "\alpha_j"] & \operatorname{colim}_j \lim_p F(p,j) \arrow[dl, dotted, "\kappa"'] \\ \operatorname{colim}_j F(p,j) & \lim_p \operatorname{colim}_j F(p,j) \arrow[l] & \end{tikzcd}$$

The left vertical map is part of the colimit being a cocone. The middle vertical map α_j then arises by functoriality of the limit. Then, κ exists because of the universal property of the upper right colimit.

For example, let $P = J = \{0,1\}$ be discrete categories and suppose all the limits and colimits exist. The map is then

$$\kappa : (A_0 \times B_0) \sqcup (A_1 \times B_1) \to (A_0 \sqcup A_1) \times (B_0 \sqcup B_1).$$

For $C = \mathrm{Set}$, this map is not an isomorphism. For $C = \mathrm{Ab}$, it is an isomorphism since in this case the finite coproducts are also products, cf. Exercise 4.8, and coproducts commute with each other.

The following statement gives a broad range of cases when the map is an isomorphism.

Proposition 4.45. Let $F : P \times J \to \mathrm{Set}$ be a functor where P is *finite* and J is filtered. Then the above canonical arrow κ is an isomorphism.

Proof. Recall we are to prove that the map

$$\kappa : \operatorname{colim}_j \lim_p F(p,j) \to \lim_p \operatorname{colim}_j F(p,j)$$

discussed above is an isomorphism, i.e., a bijection.

For surjectivity of κ, let $x \in \lim_p \operatorname{colim}_j F(p,j)$. The components $x_p \in \operatorname{colim}_J F(p,-)$ come from some $x_{p,j} \in F(p,j)$. Using the finiteness of P and the filteredness of J, we can assume that j is independent of p. The $x_{p,j}$ are not yet (in general) an element in $\operatorname{colim}_j \lim_p F(p,j)$ since, for $\alpha : p \to p'$ in P, the image of $x_{p,j}$ in $F(p',j)$ need not equal $x_{p',j}$. However, they do agree in $\operatorname{colim}_J F(p',-)$, hence there is some morphism $\beta : j \to j'$ in J such that these two elements agree in $F_{p',j'}$. We can apply this argument to each morphism of P (of which there are finitely many by assumption!), and get a $j \to j''$ such that the $x_{p,j''}$ forms an element $x_{j''} \in \lim_p F(-,j'')$. The image of $x_{j''}$ in $\operatorname{colim}_P \lim_J F$ maps to our given element $x \in \lim_J \operatorname{colim}_P F$. Thus κ is surjective.

We show the injectivity of κ: let $y, z \in \operatorname{colim}_J \lim_P D$ have the same image under κ. We have some $j \in J$ abd $y_j, z_j \in \lim_P F(-,j)$ mapping to y and z. Their images $y_{p,j}$, $z_{p,j} \in F(p,j)$ may not be equal, but they do have the same image in $\operatorname{colim}_J F(p,-)$. Hence there is some $j \to j'$ such that $y_{p,j}$ and $z_{p,j}$ map to the same element in $F(p,j')$. We repeat this for each p and get $j \to j''$ such that $y_{p,j''} = z_{p,j''}$ for all p. Hence $y_{j''} = z_{j''} \in \lim_P F(-,j'')$ so that $y = z$. $\square$

Remark 4.46. We refer to this result by saying that in Set, finite limits commute with filtered colimits. We emphasize that this contains three important specifications: that the category in question be Set, the limit is finite, and the colimit is filtered.

- An immediate extension to other categories than Set is given in Exercise 4.13. We will extend this result to a broad range of categories called locally presentable categories such as Grp, Mod_R, $\mathrm{Ban}_{\leqslant 1}$.
- For $C = \mathrm{Set}$, *some* restriction on the type of (co)limits considered is necessary, as the above discussion shows. Another related exchange statement is that so-called *sifted colimits* commute with finite products in Set. Sifted colimits, i.e., colimits of diagrams $D : J \to C$ where J is a *sifted category* are more general than filtered colimits. An example of a sifted, but non-filtered category is the *reflexive coequalizer* category

$$\begin{tikzcd} 1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & 0 \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

(two objects, three non-identity morphisms and $f \circ \sigma = g \circ \sigma = \mathrm{id}_0$.) See, e.g., [**AdamekRosickyVitale:What**
for an exposition of sifted categories and sifted colimits.

We now show that finite colimits of compact objects are again compact.

Corollary 4.47. Let C be a category with all colimits. Let $D : J \to C$ be a finite diagram (i.e., J is a finite category, Definition 4.36) such that $D(j)$ is a compact object for each object j in J. Then $\operatorname{colim} D$ is a compact object.

Proof. Let $T : I \to C$ be a filtered diagram. We have the following commutative diagram, in which the isomorphisms hold as indicated. Thus the top horizontal map is an isomorphism, so that $\operatorname{colim}_j D_j$ is compact:

$$\begin{tikzcd}
\operatorname{colim}_i \operatorname{Hom}(\operatorname{colim}_j D_j, T_i) \arrow[r] \arrow[dd, "{\cong,\text{Lemma 4.30}}"'] & \operatorname{Hom}(\operatorname{colim} D_j, \operatorname{colim} T_i) \arrow[d, "{\cong,\text{Lemma 4.30}}"] \\
 & \lim_j \operatorname{Hom}(D_j, \operatorname{colim} T_i) \\
\operatorname{colim}_i \lim_j \operatorname{Hom}(D_j, T_i) \arrow[r, "{\cong,\text{Proposition 4.45}}"'] & \lim_j \operatorname{colim}_i \operatorname{Hom}(D_j, T_i). \arrow[u, "{\cong,\text{Definition 4.42}}"']
\end{tikzcd}$$

Example 4.48. Continuing our list of compact objects in various categories, let R be a ring. An object $M \in \mathrm{Mod}_R$ is compact iff it is finitely presented (in the usual sense of algebra, i.e., the cokernel of an appropriate map $R^n \to R^m$ with $n, m < \infty$). The proof proceeds along similar lines as above: $R \in \mathrm{Mod}_R$ is compact since

$$\mathrm{Hom}_{\mathrm{Mod}_R}(R, -) = U$$

is the forgetful functor $\mathrm{Mod}_R \to \mathrm{Set}$, which does preserve filtered colimits(!) . We invoke Corollary 4.47 to see that finitely presented modules are compact.

Conversely, suppose M is compact. We have

$$M = \operatorname{colim}_{N \subset M \text{ finitely generated}} N.$$

The indexing category consisting of finitely generated submodules of M is filtered since the sum $N + N' \subset M$ is finitely generated if N and N' are. Thus, id_M factors over a finitely generated submodule N, making M a direct summand of N. Thus M is finitely generated.

Then, for a finitely generated module $M = R^n/E$, where $E \subset R^n$ is the submodule of relations. Again we have $E = \operatorname{colim}_{F \subset E \text{ finitely generated}}$ and we obtain(!) $M = \operatorname{colim}_F R^n/F$. Again id_M factors over some finitely presented module R^n/F, and hence M is itself finitely presented.

4.10 Final functors

A typical argument in elementary analysis is to consider only subsequences $(a_{n_k})_{k \in \mathbf{N}}$ instead of an originally given sequence $(a_n)_{n \in \mathbf{N}}$. In category theory, the corresponding concept is called (co)final functors. These facilitate the computation of (co)limits: given a diagram

$$D : J \to C$$

the (co)limit $\lim D$ or $\operatorname{colim} D$ can often be computed by replacing J by a smaller, hopefully simpler subcategory $I \subset J$. Instead of just using full subcategories, it is useful to go "all-in", by considering arbitrary functors $I \to J$. Given a functor $D : J \to C$, there is a natural map

$$\lim D \to \lim(D \circ K).$$

Indeed, $\lim D$ maps (compatibly) to all D_j, and therefore in particular to all $D_{K(i)}$.

Definition 4.49. A functor $K : I \to J$ between small categories I and J is called *initial* if for each functor $D : J \to C$ (with C being arbitrary), the natural map

$$\lim D \to \lim D \circ K$$

is an isomorphism.

Dually, K is called *final* if $K^{\mathrm{op}} : I^{\mathrm{op}} \to J^{\mathrm{op}}$ is initial. Equivalently, for each functor $D : J \to C$, the natural map

$$\operatorname{colim} D \circ K \to \operatorname{colim} D$$

is an isomorphism.

In order to exhibit a criterion that allows to check whether some given K is final, we need to introduce some more terminology.

Definition 4.50. Let $F : C \to D$ be a functor and $X \in D$. The *comma category*

$$F/X$$

is the category whose objects are pairs (Y, f) consisting of an object $Y \in C$ and a morphism (in D) $f : F(Y) \to X$. We define $\mathrm{Hom}_{X/F}((Y, f), (Y', f'))$ to be the set of morphisms $y : Y \to Y'$ in C such that the diagram

$$\begin{tikzcd} F(Y) \arrow[rr, "F(y)"] \arrow[dr, "f'"'] & & F(Y') \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

commutes. (The composition is the obvious one, coming from composing morphisms in C.)

Example 4.51. A dual variant of this is the comma category X/F whose objects are pairs (Y, f) with $f : F(Y) \to X$ (as opposed to $X \to F(Y)$).

A special case is $F = \mathrm{id}_C$. The categories X/id_C (resp. id_C/X) are commonly denoted X/C (resp. C/X) and called the *under-category* (resp. *over-category*) of X in C.

Concrete examples include the following:

- Given a commutative ring k, the category k/CRing is the category of k-algebras and k-algebra homomorphisms.
- In algebraic geometry, the category Sch/S, for a scheme S is the category of S-schemes.
- The category $\{*\}/\mathrm{Top}$ is the category of pointed topological spaces and base-point preserving maps.

Definition 4.52. For a small category C, the set of *path components*, denoted $\pi_0 C$ is defined as

$$\mathrm{Obj}(C)/\sim,$$

where $\sim$ is the equivalence relation generated by the relation $\sim_0$: $X \sim_0 Y$ iff there is a morphism $X \to Y$. Concretely, $X \sim Y$ if there is a finite zig-zag of morphisms in C:

$$X =: T_0 \leftarrow T_1 \to T_2 \leftarrow T_3 \dots T_n = Y.$$

A category is called *connected* if $X \sim Y$ for any two objects, i.e., $\pi_0 C$ is either empty or a singleton.

Example 4.53.
- If C has an initial or, alternatively, a final object, then $\pi_0 C = \{*\}$.
- The category Fields has $\pi_0(\mathrm{Fields}) = \{p \text{ prime }\} \sqcup \{0\}$, since there a zig-zag of morphisms between two fields precisely iff their characteristic is the same.

Proposition 4.54. Let $K : I \to J$ be a functor. The following are equivalent:

(1) for each object $j \in J$, the comma category K/j is non-empty and connected, i.e., $\pi_0(K/j) = \{*\}$,
(2) $K : I \to J$ is initial.

Example 4.55. This statement recovers Exercise 4.3: if J has an initial object e, then the inclusion

$$K : \{e\} \subset J$$

is an initial functor, i.e., for any diagram $D : J \to C$

$$\lim D = D(e).$$

Indeed, for each $j \in J$, the under-category K/j has as objects maps $e \to j$. There is precisely one such object, in particular the category is non-empty and connected.

Proof. (of Proposition 4.54) (1) $\Rightarrow$ (2): Let $D : J \to C$ be a functor. Restricting a cone over D to a cone over $D \circ K$ defines a functor

$$\mathrm{Cone}(D) \to \mathrm{Cone}(D \circ K).$$

We will show this is an isomorphism of categories, so that in particular one has a terminal object iff the other has one (i.e., $\lim D$ exists iff $\lim DK$ exists, and in that event they are the same).

Let us be given a cone X for the diagram $D \circ K$. In order to construct $X \to D(j)$ for each j, we choose some $i \in I$ and a map $K(i) \stackrel{u}{\to} j$ (which is possible since K/j is non-empty). We define a map $X \to D(j)$ as the composite

$$X \to DK(i) \stackrel{D(u)}{\to} D(i).$$

For another choice (i', u') as in the diagram below, we the composite

$$X \to DK(i') \stackrel{D(u')}{\to} D(i)$$

agrees with the previous map. By induction it suffices to consider two-step zig-zags as in the diagram:

$$\begin{tikzcd}
i \arrow[d] & \rightsquigarrow & K(i) \arrow[d] \arrow[dr, "u"] & & \rightsquigarrow & & DK(i) \arrow[d] \arrow[dr, "u"] & & . \\
i' & & K(i'') \arrow[r, "u''"] & j & & X \arrow[ur] \arrow[r] \arrow[dr] \arrow[d] & DK(i'') \arrow[r, "u''"] & Dj & \\
i' \arrow[u] & & K(i') \arrow[u] \arrow[ur, "u'"'] & & & Y \arrow[uur, dotted] \arrow[ur, dotted] \arrow[r, dotted] & DK(i') \arrow[u] \arrow[ur, "u'"'] & &
\end{tikzcd}$$

(Note that in the right diagram, the left half commutes since X is a cone over FK.) One checks this is indeed an inverse function for the objects in $\mathrm{Cone}\, D$ and $\mathrm{Cone}(D \circ K)$ (!) . This bijection on the level of objects also extends to morphisms: given a map $X \to Y$ fitting into a commutative diagram including the dotted arrows above, it yields a morphism of cones over D by composition.

For the proof of the converse (2) $\Rightarrow$ (1), we refer to [**Riehl:Categorical**]. $\square$

Example 4.56. The category $\mathbf{\Delta}$ has as objects the sets $[n] := \{0, 1, \dots, n\}$, and morphisms are given by non-decreasing maps:

$$\mathrm{Hom}_{\mathbf{\Delta}}([n], [m]) = \{f : \{0, 1, \dots, n\} \to \{0, 1, \dots, m\}, f(i) \leqslant f(j) \text{ for all } i \leqslant j (\in [n])\}.$$

For example, there are two maps $[0] \to [1]$ (mapping to 0 and 1, respectively), and one map $[1] \to [0]$. We can thus consider the (non-full) subcategory

$$\iota : \{[1] \rightrightarrows [0]\} \to \mathbf{\Delta}.$$

This functor is initial.(!) Therefore limits of diagrams

$$\Delta \to C,$$

can be computed by considering the (much smaller!) diagram consisting only of the two objects [0] and [1] instead.

A sample application of this situation is as follows: let X be a topological space, and

$$U := \coprod_{i \in I} U_i \to X$$

the coproduct of a set of open subsets $U_i \subset X$ mapping to X. Consider the functor

$$R : \Delta^{\mathrm{op}} \to \mathrm{Top}/X, [n] \mapsto \prod_{\{0,1,\dots,n\}} U,$$

the product indexed by the set $[n] = \{0, 1, \dots, n\}$ in the category Top/X of topological spaces over X. (The product in this category is the fiber product over X in the category Top.) For example, $R([0]) = U$ and $R([1]) = U \times_X U$. On morphisms, R sends a map $\alpha : [n] \to [m]$ to the map $R(\alpha) : \prod_{\{0,\dots,m\}} U \to \prod_{\{0,\dots,n\}} U$ whose component for $i \in [n]$ is the composition $\prod_{\{0,\dots,m\}} U \to \prod_{\alpha(i)} U = U$. For example, there are two maps

$$\partial_i : [0] \to [1], 0 \mapsto i,$$

and $R(\partial_i) : R([1]) = U \times_X U \to R([0]) = U$ is the projection onto the two coordinates. There is one map $s : [1] \to [0]$ and $R(s) : U \to U \times_X U$ is the diagonal map.

Suppose given a presheaf on the category Top, for example a representable presheaf $F = h_T = \mathrm{Hom}_{\mathrm{Top}}(-, T) : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$. Then

$$R^{\mathrm{op}} : \Delta \to (\mathrm{Top}/X)^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}.$$

The limit of such a composite is

$$\lim \left(F(U) \overset{\leftarrow}{\rightrightarrows} F(U \times_X U) \overset{\leftarrow}{\rightrightarrows} F(U \times_X U \times_X U) \dots \right).$$

Since the inclusion $\{[1] \rightrightarrows [0]\} \to \Delta$ is initial, the limit of this infinite diagram can be computed as the limit of

$$F(U) \rightrightarrows F(U \times_X U).$$

This simplification is implicit in the definition of a *sheaf*, where one demands that the natural map

$$F(X) \to \lim(F(U) \rightrightarrows F(U \times_X U))$$

is an isomorphism. (More precisely, a sheaf on a topological space X is a presheaf F on $\mathrm{Open}(X)$ such that for any open covering $V = \bigcup_{i \in I} V_i$ of an open subset, $F(V)$ is a limit of $\prod_{i \in I} F(V_i) \rightrightarrows \prod_{i,j} F(V_i \cap V_j)$, where the two maps are restriction along $V_i \cap V_j \subset V_i$ and $\subset V_j$, respectively.)

4.11 Exercises

Exercise 4.1. Let Fields $\subset$ CRing be the full subcategory of the category of commutative rings whose objects are fields. Show this category has neither initial nor terminal objects.

Exercise 4.2. (1) Show that Fields has all filtered colimits. What are the compact objects in this category?

(2) Fix a field k and consider $\mathrm{Fields}_{k/}$, the category of *field extensions* of k. Recall that a field extension $k \subset K$ is called finite iff $\dim_k K < \infty$. Express this as a compactness condition.

(3) Express the condition of being an algebraic extension $k \subset K$ in terms of appropriate filtered colimits and an appropriate compactness condition. (Solving this requires some basic field theory.)

Exercise 4.3. Let J be a category with an initial object 0. Let $D : J \to C$ be any diagram. Show that D_0 is a limit for D.

Exercise 4.4. Let C be a category such that all equalizers and finite products exist. Show that all fiber products exist in C. Hint: express a fiber product $A \times_B C$ in terms of an appropriate equalizers involving appropriate products.

Exercise 4.5. Let C be a category and $X \in C$ an object. The *over-category* $C_{/X}$ is the category whose *objects* are maps $T \to X$ in C and whose morphisms are commutative triangles as expected. Dually, the *under-category* $C_{X/}$ has as objects maps $X \to T$.

State and prove a necessary and sufficient condition for $C_{/X}$ (dually $C_{X/}$) to have products (resp. coproducts).

Describe concretely coproducts in the category $\mathrm{Top}_* := \mathrm{Top}_{\{*\}/}$ of *pointed topological spaces*.

Exercise 4.6. Show that $\mathbf{Z}[t] \otimes_{\mathbf{Z}} \mathbf{Z}[u] = \mathbf{Z}[t, u]$ is *not* the coproduct of $\mathbf{Z}[t]$ and $\mathbf{Z}[u]$ in the category Ring of non-commutative (but associative and unital) rings.

Hint: contemplate the corepresenting functor $h^{\mathbf{Z}[t]}$ (Example 2.14). What can you say a priori about $h^{X \sqcup Y}$ for two objects X, Y in any category?

Note: the category of rings does have all small colimits, but coproducts are less simple to describe; they involve taking free associative rings.

Exercise 4.7. Let C be a category that has all filtered colimits and all finite coproducts. Show C has arbitrary (small) coproducts.

Conclude that CRing has arbitrary (small) coproducts.

Exercise 4.8. An object $X \in C$ is called a *zero object* if it is both initial and terminal. Assume that C

- has a zero object and
- has finite coproducts and finite products
- the natural map $X \sqcup Y \to X \times Y$ (from the coproduct to the product) of two arbitrary objects $X, Y \in C$ is an isomorphism.

(1) Show that $\mathrm{Hom}_C(X, Y)$ is naturally an *abelian monoid*.

(2) C is called *additive* if it satisfies the above three conditions and if the abelian monoid $\mathrm{Hom}_C(X, Y)$ is actually an abelian group.

Which of the categories Set, Mon, Ab, Vect_k, Ban and $\mathrm{Ban}_{\leqslant 1}$ are additive?

Exercise 4.9. Let $(V_i)_{i \in I}$ be a (set-indexed) family of Banach spaces. Show that a coproduct in $\mathrm{Ban}_{\leqslant 1}$ is given by

$$\coprod V_i = \{v = (v_i) | v_i \in V_i, |v|_1 := \sum |v_i| < \infty\}$$

(and the $| - |_1$-norm). Show that a product in $\mathrm{Ban}_{\leqslant 1}$ is given by

$$\prod V_i = \{v = (v_i) | v_i \in V_i, |v|_\infty := \sup |v_i| < \infty\}$$

(and the $| - |_\infty$-norm). In particular, $l^1 = \coprod_{i \in \mathbf{N}} \mathbf{C}$ and $l^\infty = \prod_{i \in \mathbf{N}} \mathbf{C}$.

Also show that products and coproducts in Ban only exist if all but finitely many of the V_i are 0. (One may reduce to the case $V_i = \mathbf{C}$ for all i first.)

Exercise 4.10. Let G be a group and $BG \to \mathrm{Vect}_k$ a functor, i.e., a vector space V with a G-action. Compute the limit and the colimit of this functor explicitly.

Exercise 4.11. Show that $X \in \mathrm{Top}$ is compact in the sense of Definition 4.42 iff it is finite and discrete.

Hint: for the nontrivial direction, consider the canonical injection $X \to X \sqcup \mathbf{N} = \mathrm{colim}_n X \sqcup \{0, 1, \ldots, n-1\}$ and put a (non-trivial) topology on the spaces $X \sqcup \{0, 1, \ldots, n-1\}$ such that the colimit carries the coarse topology.

Exercise 4.12.
- Show that the categories CptHaus $\subset$ Top of compact Hausdorff spaces, respectively all topological spaces, have all small limits and that the inclusion preserves them. (This requires using some non-trivial theorem from point-set topology.)
- Discuss the pushout of the diagram (with the natural maps)

$$\begin{array}{ccc} [-1,1]\backslash\{0\} & \longrightarrow & [-1,1] \\ \downarrow & & \\ [-1,1] & & \end{array}$$

in the category of Hausdorff spaces (and continuous maps) and in the category Top.

Exercise 4.13. Let $C \to$ Set be a conservative functor that preserves filtered colimits and finite limits. Show that filtered colimits commute with finite limits, i.e., show that the conclusion of Proposition 4.45 holds for C.

Give explicit examples of categories C satisfying these conditions.

Exercise 4.14. Let C be a category. An object $P \in C$ is called *compact projective* if

$$\mathrm{Hom}_C(P, -) : C \to \mathrm{Set}$$

preserves filtered colimits and reflexive coequalizers. (Equivalently, using terminology that we have not yet introduced, it preserves sifted colimits. See, e.g., [**AdamekRosickyVitale:What**].)

Show that an object $P \in \mathrm{Mod}_R$ is compact projective iff it is a finitely generated projective module (i.e., there is a surjection $R^n \to P$ which splits).

Exercise 4.15. We say that a functor $F : C \to D$ *detects limits* (or *reflects limits*) if for any diagram $K : J \to C$ and any cone $\tilde{K} : J^{\lhd} \to C$ the following holds: if the composite $F \circ \tilde{K} : J^{\lhd} \to D$ is a limit cone of $F \circ K$, then $\tilde{K}$ is a limit cone for K. (Compare this to preservation of limits as in Definition 4.28.)

Show that F detects limits in the following two situations:

- C has all limits, F preserves these and F is conservative,
- F is fully faithful.

Give examples of functors where this argument can be applied.

Exercise 4.16. We say that a functor *creates limits* if it preserves (Definition 4.28) and detects (Exercise 4.15) them.

Show that the Yoneda embedding

$$y : C \to \widehat{C}$$

creates all limits.

Give an example of a category C where y does not preserve coproducts.

Exercise 4.17. Show that Set is not equivalent to $\mathrm{Set}^{\mathrm{op}}$.

Exercise 4.18. A category C is called *sifted* if it is non-empty and the diagonal functor

$$\Delta : C \to C \times C, X \mapsto (X, X)$$

is final. Show that every filtered category is sifted.
Note: The category $\mathbf{\Delta}^{\mathrm{op}}$ and its full subcategory

$$\{[0] \overset{\leftarrow}{\rightrightarrows} [1]\}^{\mathrm{op}}$$

are sifted, but not filtered. (A bare-hands proof of this assertion is tedious.) Colimits indexed by this latter category are known as *reflexive coequalizers*.

Exercise 4.19. Let R be a commutative ring and M an R-module. Let $S \subset R$ be a submonoid with respect to the multiplication (also known as a multiplicatively closed nonempty subset). In this context there is the notion of localization of R-modules [**StacksProject**].

(1) Show that the localization $M[S^{-1}]$ is a filtered colimit of an appropriate diagram

$$D : J \to \mathrm{Mod}_R$$

that has the property that $D(j) = M$ for each object $j \in D$. (Stated slightly colloquially: what are the transition maps so that $M[S^{-1}] = \mathrm{colim}(M \xrightarrow{?} M \xrightarrow{?} M \dots ?)$)

(2) Show that localization preserve finite intersections: for $M_1, \dots, M_n \subset M$, we have

$$(\bigcap_i M_i)[S^{-1}] = \bigcap_i (M_i[S^{-1}]).$$

(3) Show by an example that the localization functor $M \mapsto M[S^{-1}]$ does not preserve infinite products.

Exercise 4.20. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\star"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\star\star"] & C \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \end{tikzcd}$$

be a commutative diagram in some category. Prove:

(1) If both squares $\star$ and $\star\star$ are cartesian, then so is the total square (consisting of A, C, A' and C').

(2) If the total square and $\star\star$ are cartesian then so is $\star$.

Exercise 4.21. Describe the coequalizer of a diagram of sets,

$$R \underset{f'}{\overset{f}{\rightrightarrows}} X,$$

in terms of the following relation $\sim$ on X:

$$x \sim x' \Leftrightarrow \exists r (\in R), x = f(r), x' = f'(r).$$

Exercise 4.22. An abelian group A is called a *torsion group* if for each $a \in A$ there is some $n \in \mathbf{N}$, $n > 0$ such that $na = 0$. For an abelian group A, show that the following are equivalent:

(1) A is a torsion group,

(2) A is a filtered colimit of finite abelian groups.

Exercise 4.23. Let

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"] & A \arrow[d, "f"] \\ B' \arrow[r] & B \end{tikzcd}$$

be a cartesian diagram in a category C. Prove: if f is a monomorphism, then f' is also a monomorphism. Does the converse hold?

Exercise 4.24. Let A, B and C be abelian groups and $f : A \to B$, $g : B \to C$ two group homomorphisms such that $g \circ f = 0$. In this situation we say that

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a *short exact sequence* of abelian groups, if (by definition), g is surjective, $\ker g = \mathrm{im}\, f$ and f is injective.

Show that f and g fit into a short exact sequence iff the square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\
0 \arrow[r] & C
\end{tikzcd}$$

is both cartesian and cocartesian, i.e., cartesian in the opposite category $(\mathrm{Ab})^{\mathrm{op}}$.

Exercise 4.25. Let I be the christmas-tree-category: its objects are the christmas tree balls located at the endpoints of the branches, and there are morphisms as depicted (and also composites of these morphisms, not depicted):

$$\begin{tikzcd}
 & & & \bullet \arrow[d] & & & \\
 & & & \bullet \arrow[dl] \arrow[dr] & & & \\
 & & \bullet \arrow[dl] & & \bullet \arrow[dr] & & \\
 & \bullet \arrow[r] & \bullet \arrow[dl] & & \bullet \arrow[dr] & \bullet \arrow[l] & \\
 & \bullet \arrow[dl] & & & & \bullet \arrow[dr] & \\
\bullet \arrow[r] & \bullet \arrow[d] & & & & \bullet \arrow[d] & \bullet \arrow[l] \\
 & \bullet & & & & \bullet &
\end{tikzcd}$$

Solve one of the following problems:

- Let

$$K : I \to C$$

be a diagram taking values in a merry category. Compute its limit and its colimit!

Hint: find appropriate final (resp. cofinal) full subcategories of I.

- Bring self-made christmas cookies to the exercise session!

Exercise 4.26. Let $\mathbf{\Delta}$ be as in Example 4.56 and let $\Delta_n \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ be the functor represented by $[n]$. Show that for any $m, n \neq 0$ the functor

$$\Delta_n \times \Delta_m \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}; [k] \mapsto \mathrm{Hom}_{\mathbf{\Delta}}([k],[n]) \times \mathrm{Hom}_{\mathbf{\Delta}}([k],[m])$$

is not representable. Let C be the category of partially ordered sets with non decreasing maps as morphisms. Prove that the functor $F \colon C^{\mathrm{op}} \to \mathrm{Set}; X \mapsto \mathrm{Hom}(X,[n]) \times \mathrm{Hom}(X,[m])$ is representable. Use this to describe $\Delta_n \times \Delta_m$.

Exercise 4.27. A category C is called *pointed* if it has a zero object (Exercise 4.8), denoted $0 \in C$. A *zero morphism* is a morphism $f : X \to Y$ that factors over some zero object (equivalently, by Lemma 4.3, over any zero object). A zero morphism is also denoted by 0. In a pointed category C, a *kernel* (resp. *cokernel*) is a limit (resp. colimit) of a diagram

$$X \underset{0}{\overset{f}{\rightrightarrows}} Y.$$

It is denoted by $\ker f$ (resp. $\operatorname{coker} f$).

(1) Show that the category Grp of groups is pointed.

(2) Show that the kernel, in the sense defined above, of a group homomorphism $f : G \to H$ agrees with $\{g \in G, f(g) = e_H\}$ (and the natural inclusion into G).

(3) Show that the cokernel, in the sense defined above, $\operatorname{coker} f$ is the $H/N_H(f(G))$ (together with the canonical map $H \to H/N_H(f(G))$. Here $N_H(f(G))$ denotes the normalizer of the subgroup $f(G) \subset H$.

Chapter 5

Adjunctions

It was already mentioned that functors are what connects different categories. Many times, we don't like one way tickets, but want to come back at some point. Given a functor

$$F : C \to D,$$

(how) can we go back from D to C? In complete generality, we cannot expect this. In special cases, namely if F is (part of) an equivalence of categories, we do have an inverse functor. Adjunctions are in between the void and the perfect: we will have two functors $F : C \to D$ and $G : D \to C$ that are closely related, but (in general) stopping short of being an equivalence.

The situation is comparable to the situation with (continuous, linear) maps between Hilbert spaces: given such a map (a.k.a. operator) $f : H \to H'$, an *adjoint operator* is a map $g : H' \to H$ such that

$$\langle f(x), y\rangle = \langle x, g(y)\rangle.$$

In category theory, functions get replaced by functors and the inner product between vectors gets replaced by Hom-sets between objects.

We will see in §5.1 that adjunctions are all over the place. We will understand that having an adjoint functor for F tells us a lot about the functor. Notably, we will establish a close bond between (co)limits and adjoints.

Given the interest in establishing adjoint functors, we will prove an easy *adjoint functor theorem* (Theorem 5.19), which constructs, for example, the Stone-Čech compactification functor

$$\beta : \mathrm{Top} \to \mathrm{CptHaus}.$$

We will then single out and study a particular class of adjunctions, so-called *monadic adjunctions*.

5.1 Definitions and examples

Definition 5.1. Given a pair of functors

$$L : C \rightleftarrows D : R,$$

an *adjunction* is an isomorphism of functors

$$\Psi : \mathrm{Hom}_D(L-, -) \overset{\cong}{\Rightarrow} \mathrm{Hom}_C(-, R-).$$

(Both are functors $C^{\mathrm{op}} \times D \to \mathrm{Set}$.) In this event, L is called the *left adjoint*, R the *right adjoint*. We say that "L is a left adjoint", if there is some functor R for which there is such an adjunction. Similarly for saying that "R is a right adjoint". The existence of an adjunction between two functors as above is denoted by

$$L \dashv R \quad \text{or} \quad R \vdash L \quad \text{or} \quad \begin{tikzcd} C \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "R"] \end{tikzcd}.$$

In such a depiction, left adjoints will always be drawn on top of their right adjoints in these notes, but not all authors follow such a convention.

Remark 5.2. By definition of an isomorphism of functors, this means that for each $X \in C$, $Y \in D$, we have an isomorphism of sets (i.e., a bijection)

$$\Psi_{X,Y} : \mathrm{Hom}_D(LX, Y) \to \mathrm{Hom}_C(X, RY).$$

This isomorphism is functorial (a.k.a. natural) in X and Y in that for each morphism $f : X \to X'$ and $g : Y \to Y'$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_D(LX', Y) \arrow[r, "h \mapsto g \circ h \circ L(f)"] \arrow[d, "\Psi_{X',Y}"] & \mathrm{Hom}_D(LX, Y') \arrow[d, "\Psi_{X,Y'}"] \\
\mathrm{Hom}_C(X', RY) \arrow[r, "h \mapsto R(g) \circ h \circ f"] & \mathrm{Hom}_C(X, RY').
\end{tikzcd}$$

Adjunctions are everywhere, too numerous to name even a fraction! First of all, every equivalence of categories

$$F : C \rightleftarrows D : F^{-1}$$

is an adjunction. Indeed, $\mathrm{Hom}_C(-,-) \to \mathrm{Hom}_D(F(-), F(-))$ is a bijection (with inverse given by applying F^{-1}), so that we obtain a bijection, functorial in X and Y:

$$\mathrm{Hom}_D(F(X), Y) = \mathrm{Hom}_C(F^{-1}(F(X)), F^{-1}(Y)) \cong \mathrm{Hom}_C(X, F^{-1}(Y)).$$

(Note that, by definition of an equivalence, there is a *functorial* isomorphism $\mathrm{id} \xrightarrow{\cong} F^{-1} \circ F$.) As a partial converse, given an adjunction, one can construct an equivalence on certain subcategories, see Exercise 5.1.

Example 5.3. We consider the forgetful functor

$$U : \mathrm{Ab} \to \mathrm{Set}.$$

We show it admits a left adjoint, known as the *free abelian group* functor,

$$\mathbf{Z}[-] : \mathrm{Set} \to \mathrm{Ab}.$$

If this exists, it must satisfy

$$\mathrm{Hom}_{\mathrm{Ab}}(\mathbf{Z}[\{*\}], A) \overset{!}{\cong} \mathrm{Hom}_{\mathrm{Set}}(\{*\}, A) = A.$$

(!) This is(!) satisfied if we set the free abelian group on a single generator to be $\mathbf{Z}[\{*\}] := \mathbf{Z}$. Furthermore, for the free abelian group on any set S, we observe that

$$S = \coprod_{s \in S} \{*\}.$$

Thus, the free abelian group on S needs to satisfy

$$\mathrm{Hom}_{\mathrm{Ab}}(\mathbf{Z}[S], A) \overset{!}{\cong} \mathrm{Hom}_{\mathrm{Set}}(S, A) = \mathrm{Hom}_{\mathrm{Set}}(\coprod_{s \in S} \{*\}, A) = \prod_{s \in S} \mathrm{Hom}_{\mathrm{Set}}(\{*\}, A) = \prod_{s} A.$$

We do get such an isomorphism if we set

$$\mathbf{Z}[S] = \coprod_{s \in S} \mathbf{Z}.$$

More concretely, we can set

$$\mathbf{Z}[S] := \{f : S \to \mathbf{Z}, f(s) = 0 \text{ for all but finitely many } s \in S\} = \bigoplus_{s \in S} \mathbf{Z}$$

regarded as an abelian group by pointwise addition. This is functorial in S, by sending a morphism $r : S \to T$ of sets to the map of abelian groups

$$\mathbf{Z}[S] \to \mathbf{Z}[T], (f : S \to \mathbf{Z}) \mapsto (T \to \mathbf{Z}, t \mapsto \sum_{s, r(s) = t} f(s)).$$

One checks(!) this is functorial and gives an adjunction !

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "\mathbf{Z}[-]"] \arrow[r, phantom, "\perp"] & \mathrm{Ab}. \arrow[l, bend left, "U"] \end{tikzcd}$$

In view of Lemma 5.7 and Exercise 5.4, we have actually no choice (up to an isomorphism of functors) in defining this left adjoint.

The functor U does *not* preserve coproducts ($\mathbf{Z} \oplus \mathbf{Z}$, which is the coproduct in Ab, is not bijective to $\mathbf{Z} \sqcup \mathbf{Z}$, the coproduct in sets). Hence U does not admit a right adjoint by Lemma 5.7.

The discussion above goes through without any changes if we replace $\mathbf{Z}$ by any ring R, giving rise to a the free-forgetful adjunction

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[r, bend right, "U"'] & \mathrm{Set} \arrow[l, bend right, "R[-]"'] \end{tikzcd}$$

Example 5.4. For a set X, we can consider the power set $P(X)$, as a poset ordered by inclusion, and thus as a category (Example 2.9). Given a map (of sets) $f : X \to Y$, we then have an order-preserving map, hence a functor

$$f^{-1} : P(Y) \to P(X), V \mapsto f^{-1}(V).$$

This functor has a left adjoint, known as the *existential quantifier*:

$$\exists_f : P(X) \to P(Y), U \mapsto f(U) = \{y \in Y | \exists x \in f^{-1}(y), x \in U\}.$$

It also has a right adjoint, known as the *all quantifier*:

$$\forall_f : P(X) \to P(Y) : U \mapsto \{y \in Y | \forall x \in f^{-1}(y), x \in U\}.$$

Thus, we get a chain of adjunctions

$$\exists_f \dashv f^{-1} \dashv \forall_f.$$

For, say, the left adjunction we need to check

$$\exists_f U \subset V \Longleftrightarrow U \subset f^{-1}(V).$$

"$\Rightarrow$": for $x \in U$, $y := f(x) \in \exists_f U \subset V$, so $x \in f^{-1}(V)$. "$\Leftarrow$": for $y \in \exists_f U$, there is $x \mapsto y$, $x \in U \subset f^{-1}(V)$, so $y \in V$.

Example 5.5. There is a chain of adjunctions, where left adjoints are drawn on top of right adjoints: U is the forgetful functor, D is the functor equipping a set with the *discrete topology* (any subset is open), C puts the coarse topology on a set X (only $\varnothing$ and X are open),

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "U"] & \mathrm{Set} \arrow[l, bend right, "D"'] \arrow[l, bend left, "C"] \end{tikzcd}$$

Indeed, given a set X and a topological space T, we have

$$\mathrm{Hom}_{\mathrm{Set}}(X, UT) = \mathrm{Hom}_{\mathrm{Top}}(DX, T),$$

i.e., any map of sets $X \to UT$ is continuous with the discrete topology on X. The chain of adjoints can not be prolonged: if C had a right adjoint, it would need to send coproducts to coproducts (by Lemma 5.7). However, the coproduct in Top, $C(S) \sqcup C(S)$ does *not* have the coarse topology (it has 4 open subsets instead). Similarly, if D had a left adjoint, it would preserve products, so that the product topology on $\prod_{i \in \mathbf{N}} \{0, 1\}$ would need to be discrete. This is however not the case. A remedy for this is offered by Exercise 5.3.

Example 5.6. Let $f : R \to S$ be a map of (not necessarily commutative, but associative and unital) rings. As above, let Mod_S be the category of left S-modules with S-linear maps as morphisms. The forgetful functor U (sending an S-module M to the same abelian group M, but now regarded as an R-module via f) has two adjoints:

$$\begin{tikzcd}
\mathrm{Mod}_S \arrow[r, "U"] & \mathrm{Mod}_R \arrow[l, bend right, "S\otimes_R -"'] \arrow[l, bend left, "\mathrm{Hom}_R(S,-)"]
\end{tikzcd}$$

Here $\mathrm{Hom}_R(S, -)$ is the set of homomorphisms of left R-modules, regarded as a left S-module via the right action of S on itself.

5.2 Limits and adjoints

The concepts of limits and adjoints are closely related: right adjoints preserve limits (dually left adjoints preserve colimits), and limit functors *are* right adjoints (dually, colimits are left adjoints).

Lemma 5.7. Let $L : C \to E$ be a left adjoint. Then L preserves colimits in the sense that for any diagram $D : J \to C$ whose colimit exists (in C), then $L(\operatorname{colim} D)$ is a colimit of $L \circ D$. In brief:

$$L(\operatorname{colim} D) = \operatorname{colim}(L \circ D).$$

Stated slightly more colloquially,

$$L(\operatorname{colim} D_i) = \operatorname{colim} L(D_i).$$

Dually, right adjoints preserve (existing) limits.

Proof. Let us be given a cocone for $L \circ D$, i.e., an object $T \in E$ that fits into a commutative diagram like so, here $i, j, k \in J$ and $\alpha : i \to j$ stands for a map in D:

$$\begin{tikzcd}
& L(\operatorname{colim} D_i) \arrow[rr, dotted, "\exists!"] & & T \\
L(D_i) \arrow[ur, "L(p_i)"] \arrow[r, "L(\alpha)"'] \arrow[urrr] & L(D_j) \arrow[u, "L(p_j)"] \arrow[urr] & L(D_k) \arrow[ul] \arrow[ur] & \cdots \arrow[ulll] \arrow[u]
\end{tikzcd}$$

We need to show there exists a unique dotted map so that the diagram still commutes.

We now apply the adjunction isomorphism, crucially (!) also using that this isomorphism is *functorial* to see that such a diagram as above is equivalent to

$$\begin{tikzcd}
& \operatorname{colim} D_i \arrow[rr, dotted, "\exists!"] & & R(T) \\
D_i \arrow[ur, "p_i"] \arrow[r, "\alpha"'] \arrow[urrr] & D_j \arrow[u, "p_j"] \arrow[urr] & D_k \arrow[ul] \arrow[ur] & \cdots \arrow[ulll] \arrow[u]
\end{tikzcd}$$

In this diagram a dotted map exists uniquely as stated. Again by adjunction, this is equivalent to the existence and unicity of the dotted map in the previous diagram. This shows that $L(\operatorname{colim} D_i)$ is a colimit for $L \circ D$. $\square$

Example 5.8. For a map $f : X \to Y$ of sets, the preimage functor $f^{-1} : P(Y) \to P(X)$ between the power sets preserves unions and intersections. Indeed, these are colimits and limits, respectively (!) , so the claim follows from f^{-1} having a left and a right adjoint.

Example 5.9. As in Example 5.6, let $f : R \to S$ be a ring homomorphism. The functor $S \otimes_R -$ preserves all colimits. In particular, it preserves *cokernels*, i.e., coequalizers of the form

$$(M \overset{f}{\underset{0}{\rightrightarrows}} N).$$

Dually, $\mathrm{Hom}_R(S, -)$ preserves limits and, in particular, *kernels*. Indeed, they are left (resp. right) adjoint to the forgetful functor U.

Lemma 5.10. Let J be a small category and C be a category with all J-shaped limits (resp. J-shaped colimits). Then the limit (res. colimit) functor constructed in Lemma 4.26 is a right (resp. left) adjoint of the diagonal functor $\Delta : C \to \mathrm{Fun}(J, C)$ (Definition 4.25).

Proof. For $X \in C$ and a diagram $D : J \to C$, a bijection

$$\mathrm{Hom}_{\mathrm{Fun}(J,C)}(\Delta(X), D) \to \mathrm{Hom}_C(X, \lim D)$$

is obtained by noticing that a map $\Delta(X) \to D$ is nothing but a cone on D, with tipping point X. This is the same as a map $X \to \lim D$. One checks(!) that this bijection is functorial in X and D. $\square$

Lemma 5.11. Let

$$C \xrightarrow{F} D \xrightarrow{G} E$$

be two composable functors. Suppose F and G have a right adjoint $F^R : D \to C$ and $G^R : E \to D$, respectively. Then $F^R \circ G^R$ is a right adjoint of $G \circ F$.

Proof. This follows from composing the bijections

$$\mathrm{Hom}_C(-, (F^R \circ G^R)(-)) \cong \mathrm{Hom}_D(F(-), G^R(-)) \cong \mathrm{Hom}_E((G \circ F)(-), -).$$

Example 5.12. Dually, the same also applies for the composite of left adjoints. For example, let $R \to S \to T$ be two ring homomorphisms. The composite of the forgetful functors $\mathrm{Mod}_T \to \mathrm{Mod}_S \to \mathrm{Mod}_R$ is again the forgetful functor. Thus, passing to left adjoints, we obtain functorial isomorphisms:

$$T \otimes_S S \otimes_R \otimes - \cong T \otimes_R -.$$

Now we can (conveniently) prove the categorical Fubini theorem, stated in Lemma 4.44.

Proof. We want to show that for $F : I \times J \to C$, the object $X := \lim_i \lim_j F(i, j)$ is, provided that all these limits exist, a limit of F. We use the Yoneda embedding $y : C \to \widehat{C}$, which creates (i.e., preserves and detects) limits (Exercise 4.16). Thus X is a limit for F iff $y(X)$ is a limit for $y \circ F$ and $y(X) = \lim_i \lim_j y(F(i, j))$. We can therefore replace our functor F by $I \times J \to C \xrightarrow{y} \widehat{C}$. The category $\widehat{C}$ has all (small) limits (Corollary 4.35), so we may assume C has all limits.

As outlined in the proof sketch of Lemma 4.44, the diagram involving the diagonal functors

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\Delta"] & \mathrm{Fun}(I, C) \arrow[d, "\Delta"] \\
\mathrm{Fun}(J, C) \arrow[r, "\Delta"] & \mathrm{Fun}(I \times J, \Delta)
\end{tikzcd}$$

obviously commutes. Thus, the respective limit functors, also commute. $\square$

5.3 The triangle identities

Instead of the definition given above, adjunctions can also be approached in a different (but ultimately equivalent) way, as follows.

Definition 5.13. Given an adjunction

$$L : C \rightleftarrows D : R$$

we call a morphism $f : X \to R(Y)$ (in C) and a morphism $g : L(X) \to Y$ *transposes* if they correspond to each other under the bijection

$$\Psi_{X,Y} : \mathrm{Hom}_C(L(X), Y) \cong \mathrm{Hom}_D(X, R(Y)).$$

Lemma 5.14. Let

$$L : C \rightleftarrows D : R$$

be two functors.

Then there is an adjunction between L and R (i.e., a functorial isomorphism Ψ as in Definition 5.1 above) if and only if there is a pair of natural transformations

$$\eta : \mathrm{id}_C \Rightarrow RL \text{ and } \epsilon : LR \Rightarrow \mathrm{id}_D$$

satisfying the so-called *triangle identities*

$$\begin{tikzcd} L \arrow[r, Rightarrow, "L\eta"] \arrow[dr, equal, "\mathrm{id}"'] & LRL \arrow[d, Rightarrow, "\epsilon L"] \\ & L, \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, Rightarrow, "\eta R"] \arrow[dr, equal, "\mathrm{id}"'] & RLR \arrow[d, Rightarrow, "R\epsilon"] \\ & R. \end{tikzcd}$$

The transformation η is then called the *unit* and ϵ is called the *counit* of the adjunction.

Proof. "$\Rightarrow$": for an object $X \in C$, $\eta_X : X \to RLX$ is defined to be the transpose of $\mathrm{id}_{L(X)} : L(X) \to L(X)$, and $\epsilon_Y : LRY \to Y$ the transpose of id_{RY}. The statement that η is natural amounts to the commutativity of the left hand square below, which is (using the *functoriality* of Ψ!), equivalent to the commutativity of the right hand square, which is obvious:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"] & RLX \arrow[d, "RLf"] \\ X' \arrow[r, "\eta_{X'}"] & RLX' \end{tikzcd} \qquad \begin{tikzcd} LX \arrow[r, "\mathrm{id}_{LX}"] \arrow[d, "Lf"] & LX \arrow[d, "Lf"] \\ LX' \arrow[r, "\mathrm{id}_{LX'}"] & LX'. \end{tikzcd}$$

The left triangle identity is equivalent to the commutativity of the left square below, which again by adjunction is equivalent to the commutativity of the right square below, which is obvious:

$$\begin{tikzcd} LX \arrow[r, "\mathrm{id}_{LX}"] \arrow[d, "L\eta_X"] & LX \arrow[d, "\mathrm{id}_{LX}"] \\ LRLX \arrow[r, "\epsilon_{LX}"] & LX \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\eta_X"] & RLX \arrow[d, "\mathrm{id}_{RLX}"] \\ RLX \arrow[r, "\mathrm{id}_{RLX}"] & RLX. \end{tikzcd}$$

"$\Leftarrow$": Given ϵ and η, we define two maps

$$\mathrm{Hom}_D(LX, Y) \to \mathrm{Hom}_C(X, RY), (f : LX \to Y) \mapsto (X \stackrel{\eta_X}{\to} RLX \stackrel{R(f)}{\to} RY),$$

and conversely

$$\mathrm{Hom}_C(X, RY) \to \mathrm{Hom}_D(LX, Y), (g : X \to RY) \mapsto (LX \stackrel{Lg}{\to} LRY \stackrel{\epsilon_Y}{\to} Y).$$

Using the triangle identities, one checks(!) that these two maps are indeed mutual inverses. □

5.4 Constructing adjoint functors via the solution set condition

We know that right adjoints preserve limits (Lemma 5.7). *Adjoint functor theorems* are theorems asserting that – under appropriate further hypotheses on the involved categories – limit-preserving functors are right adjoint. In this section, we develop the most basic adjoint functor theorem, using Freyd's solution set condition.

As a special case, consider the unique functor

$$P : C \to \{*\}$$

sending all objects (and morphisms) in some category C to $*$ (and id_*). A left adjoint L of this functor, if it exists, is an object $X := L(*) \in C$ such that for all $Y \in C$:

$$\mathrm{Hom}_C(X, Y) = \mathrm{Hom}_{\{*\}}(*, p(Y)) = \mathrm{Hom}_{\{*\}}(*, *) = \{\mathrm{id}_*\}.$$

In other words, X needs to be an initial object. The statement to come can therefore be regarded as asserting the existence of a left adjoint to some (very) special functors.

Definition 5.15. A set $\{X_i\}_{i \in I}$ of objects in a category C is called *weakly initial* if for each $Y \in C$ there is some $i \in I$ and a map $X_i \to Y$.

The point in this definition is that we have a *set* of such objects. If we were to allow a class at this point, the definition would be vacuous: in this case we could take the class of all objects in C, and the identity morphisms.

Proposition 5.16. Let C be a category that is complete, i.e., has all small limits. Then C has an initial object iff it satisfies the following condition, known as the *solution set condition*: there is a weakly initial set $\{X_i\}_{i \in I}$.

Proof. If C has an initial object 1, then the solution set condition is satisfied by definition of an initial object.

Conversely, assume we have a weakly initial set $\{X_i\}_{i \in I}$. Since C has all small (i.e., set-indexed) limits, we can consider $W := \prod_{i \in I} X_i$. For each $Y \in C$ there is at least one morphism $W \to Y$, namely $W = \prod X_i \to X_i \to Y$. We can further consider

$$E := \mathrm{eq}(W \rightrightarrows \prod_{f \in \mathrm{End}_C(W)} W) \xrightarrow{i} W,$$

where the projection to the copy corresponding to f are the maps id_W and f, respectively. We claim that E is an initial object. For each $Y \in C$ there is a map $E \to W \to Y$.

Suppose there are two maps $f, g : E \to Y$. We will show $f = g$ which shows that E is initial. Consider the diagram:

$$\begin{tikzcd}
& U := \mathrm{eq}(f,g) \arrow[r, "\iota"] & E \arrow[r, "f,g"] \arrow[d, "i"] & Y \\
E \arrow[r, "i"] \arrow[urr, bend left, equal] & W \arrow[u, dotted, "s"] \arrow[r, shift left, "\iota s"] \arrow[r, shift right, "\mathrm{id}_W"'] & W.
\end{tikzcd}$$

By the above property of W, there is *some* morphism $s : W \to U$ as indicated. We have two endomorphisms of W, the composite $i \circ \iota \circ s$, and id_W. By construction of E, $(i \circ \iota \circ s)i = \mathrm{id}_W \circ i = i \circ \mathrm{id}_E$. Being an equalizer map, i is a monomorphism (Lemma 4.20), so that

$$\iota \circ s \circ i = \mathrm{id}_E,$$

i.e., the part in the diagram containing thge curved "=" arrow and $\iota s i$ commutes. Thus ι has a right inverse. Again, ι being an equalizer is a monomorphism. Having a right inverse means it is an isomorphism so that $f = g$. □

We intend to upgrade this statement to the existence of more general left adjoints. Before stating that adjoint functor theorem we show that it is enough to establish the value of a would-be left adjoint L on individual objects.

Lemma 5.17. Let $R : D \to C$ be a functor such that for each $X \in C$ the functor

$$D \to \mathrm{Set}, Z \mapsto \mathrm{Hom}_C(X, R(Z)) \tag{5.18}$$

is corepresentable by some object $L(X)$. (I.e., there is a bijection, functorial in Z, $\mathrm{Hom}_D(L(X), Z) = \mathrm{Hom}_C(X, R(Z))$.) Then there is a unique functor

$$L : C \to D$$

which is given on objects by $X \mapsto L(X)$ and such that the family of isomorphisms in (5.18) is also functorial in X.

Proof. This lemma is very similar to Lemma 4.26 and can be proven as outlined in Exercise 5.11. Alternatively, one may use the Yoneda lemma in the guise of the equivalence

$$y : D \stackrel{\cong}{\to} \mathrm{Fun}(D, \mathrm{Set})^{\mathrm{corepr}},$$

where at the right we have the full subcategory of the functor category consisting of corepresentable functors (Lemma 3.6). Then define L to be the composite

$$C \xrightarrow{c \mapsto \mathrm{Hom}_C(c, R(-))} \mathrm{Fun}(D, \mathrm{Set})^{\mathrm{corepr}} \stackrel{y^{-1}}{\to} D.$$

One checks this is a left adjoint to R. ☐

Theorem 5.19. (Adjoint functor theorem via the solution set condition) Let $R : D \to C$ be a functor that preserves limits. We assume:
(1) D is complete, i.e., has all limits.

(2) For each $c \in C$, the comma category (Example 4.51) c/R has a weakly initial set of objects.
Then R has a left adjoint (i.e., is a right adjoint).

Remark 5.20. Condition (2) amounts to the following *solution set condition*: for each $c \in C$ there is a (small) set I and a family of maps $f_i : c \to R(d_i)$ such that any arrow $h : c \to R(d)$ can be written as a composite for some i and some map $t : d_i \to d$

$$\begin{tikzcd} c \arrow[r, "f_i"] \arrow[dr, "h"'] & R(d_i) \arrow[d, "R(t_i)"] \\ & R(d). \end{tikzcd}$$

Roughly speaking, this means that the class of maps out of c into $R(d)$, which is potentially a class of morphisms, is actually "controlled" by a set of such morphisms.

Proof. We break the proof into three steps:
(1) R admits a left adjoint iff the comma category c/R has an initial object for each $c \in C$.

(2) As D is complete and R preserves limits, the comma category c/R is complete.

(3) Hence, by Proposition 5.16, c/R has an initial object. ☐

Lemma 5.21. A functor $R : D \to C$ admits a left adjoint iff the comma category c/R has an initial object for each $c \in C$.

Proof. "$\Leftarrow$": By Lemma 5.17, we have to show that for each $c \in C$ functor

$$\mathrm{Hom}_C(c, R(-)) : D \to \mathrm{Set}$$

is corepresentable. Picking an initial object $(d, f : c \to R(d))$ in the comma category, we construct a functorial bijection

$$\mathrm{Hom}_D(c, R(d')) \stackrel{1:1}{\rightleftarrows} \mathrm{Hom}_D(d, d').$$

The two maps are constructed by glancing at the following diagram.

$$\begin{tikzcd} c \arrow[r] \arrow[dr] & R(d) \arrow[d, "R(\delta)"] \\ & R(d'). \end{tikzcd}$$

A map $\delta : d \to d'$ is sent to the diagonal composite. Conversely, a map $c \to R(d')$ is an object in the comma category, so there is a (unique) map δ making this diagram commute. The resulting maps are indeed inverse by the unicity.

"$\Rightarrow$" is left as an exercise. $\square$

Lemma 5.22. Suppose $R : D \to C$ is a limit-preserving functor and D is complete. Then the comma category c/R is complete for each $c \in C$.

Proof. Let $K : I \to c/R$ be a diagram, written as $(d_i, f_i : c \to R(d_i))$. We can pick a limit $d = \lim d_i$. From the f_i, we get a map $c \to \lim R(d_i)$, which yields a map $c \to R(d)$ since R preserves limits, i.e., $R(\lim d_i) \xrightarrow{\cong} \lim R(d_i)$. One checks that this object is indeed a limit for K. $\square$

Corollary 5.23. The forgetful functor

$$U : \mathrm{Grp} \to \mathrm{Set}$$

has a left adjoint, the *free group functor*.

Proof. We know that Grp has all limits and that U preserves them. Indeed, it suffices to check this for products and equalizers (Proposition 4.32), both of which are readily checked(!) . We now check the solution set condition: fix $X \in \mathrm{Set}$. Consider a map (of sets) $f : X \to UG$. Let $S \subset G$ be the subgroup of G generated by the elements $f(x)$, $x \in X$. Then every element of S is a finite product of the form $(f(x_1))^{\pm 1} \dots (f(x_n))^{\pm 1}$. Thus the cardinality of S is bounded (by $\aleph_0 \cdot \sharp X$). In the class of groups S arising in this way (for all groups G and all maps f), pick one element of each isomorphism class. Since the cardinality of the groups arising in this way is bounded, there is a *set* of such isomorphism classes. The collection of all maps $X \to U(S)$, with S in this set, forms a set, and is a solution set. $\square$

We will now construct the tensor product

$$M \otimes_R N \in \mathrm{Ab}$$

of a right R-module M and a left R-module N. Recall that the desired universal property of such a tensor product is that

$$\mathrm{Hom}_{\mathrm{Ab}}(M \otimes_R N, T) = \{f : M \times N \to T, \text{ additive in each variable separately}, f(m \cdot r, n) = f(m, r \cdot n)\}.$$

The maps at the right hand side are called *balanced* maps.

Corollary 5.24. Let R be a ring, M a right R-module and N a left R-module. Then the functor

$$F : \mathrm{Ab} \to \mathrm{Set}, T \mapsto \{f : M \times N \to T, \text{ balanced}\}$$

is corepresentable.

Proof. The functor preserves limits, which is again quickly checked for products and equalizers and then one uses Proposition 4.32. Taking our cue from the proof of Corollary 5.23, we consider a balanced map $f : M \times N \to T$ and consider the subgroup $T' \subset T$ generated by the elements $f(m, n)$ for all $m \in M$, $n \in N$. Its cardinality is again bounded (by $\aleph_0 \cdot \sharp M \cdot \sharp N$). So, up to isomorphism there is only a *set* of such subgroups T'. We obtain a solution set for our functor since f factors as a balanced map (i.e., an element in $F(T')$) followed by the inclusion $T' \subset T$. $\square$

Corollary 5.25. There is a functor

$$- \otimes_{\mathbf{Z}} - : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$$

such that for each $M \in \mathrm{Ab}$, there are adjunctions

$$M \otimes_{\mathbf{Z}} - : \mathrm{Ab} \rightleftarrows \mathrm{Ab} : \mathrm{Hom}(M, -),$$

$$- \otimes_{\mathbf{Z}} M : \mathrm{Ab} \rightleftarrows \mathrm{Ab} : \mathrm{Hom}(M, -).$$

Proof. We choose a corepresenting object, denoted $M \otimes_{\mathbf{Z}} N$, of the above functor. This can be made into a functor as in Lemma 5.17. The stated adjunctions then hold by design: e.g.

$$\mathrm{Hom}(M \otimes_{\mathbf{Z}} N, K) = \{\text{balanced maps } f : M \times N \to K\} = \mathrm{Hom}(N, \mathrm{Hom}(M, K)).$$

The triple of functors $- \otimes_{\mathbf{Z}} -$, $\mathrm{Hom}(-, -)$, $\mathrm{Hom}(-, -)$ is an example of a two-variable adjunction in the following sense:

Definition 5.26. Given three categories, A, B and C, a *two-variable adjunction* is a triple of functors

$$F : A \times B \to C, G : A^{\mathrm{op}} \times C \to B, H : B^{\mathrm{op}} \times C \to A$$

together with functorial bijections (for $a \in A$, $b \in B$ and $c \in C$)

$$\mathrm{Hom}_C(F(a, b), c) = \mathrm{Hom}_B(b, G(a, c)) = \mathrm{Hom}_A(a, H(b, c)).$$

Example 5.27. Let $A = B = C = \mathrm{Set}$, and let $F : \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$ be the product. This functor is part of a two-variable adjunction with

$$G : \mathrm{Set}^{\mathrm{op}} \times \mathrm{Set} \to \mathrm{Set}, (a, c) \mapsto \mathrm{Hom}(a, c),$$

$$H : \mathrm{Set}^{\mathrm{op}} \times \mathrm{Set} \to \mathrm{Set}, (b, c) \mapsto \mathrm{Hom}(b, c).$$

Indeed, the above bijections, sometimes referred to as *currying*, are then natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Set}}(a \times b, c) &= \mathrm{Hom}_{\mathrm{Set}}(a, \mathrm{Hom}(b, c)) \\ f &\mapsto (\alpha \mapsto (\beta \mapsto f(\alpha, \beta))) \\ ((\alpha, \beta) \mapsto g(a)(b)) &\leftarrow\!\shortmid g, \end{aligned}$$

and similarly with $\mathrm{Hom}(a, c)$.

We give a final application of the adjoint functor theorem.

Corollary 5.28. The forgetful functor

$$U : \mathrm{CptHaus} \to \mathrm{Top}$$

admits a left adjoint, commonly denoted by β and called the *Stone-Čech compactification*.

Proof. To check that a fully faithful functor $C \subset D$, such as $\mathrm{CptHaus} \subset \mathrm{Top}$, preserves limits, it is enough to see that a product and equalizer in the category D, of a diagram that only takes values in the subcategory C, is an object of C. In the present situation *Tychonoff's theorem* asserts this for products. Equalizers of compact Hausdorff spaces are closed subspaces of compact Hausdorff spaces, and therefore again compact Hausdorff.

In order to check the solution set condition we make the following claim: given a compact Hausdorff space K and a subset $T \subset K$ which is dense (i.e., $\overline{T} = K$; throughout closures are taken in K), we have

$$\sharp K \leqslant 2^{2^{\sharp T}}.$$

We prove this claim by constructing an injective map:

$$L : K \to P(P(T)), k \mapsto \{V \subset T, k \in \overline{V}\}$$

into the double power set. We show L is injective: given $k \neq k'$ we can find (since K is Hausdorff) open neighborhoods $U \ni k$, $U' \ni k'$ such that $U \cap U' = \varnothing$. Then $\overline{U \cap T} \cap U' \subset \overline{U} \cap U' = \varnothing$ (since both are open), so that $k' \notin \overline{U \cap T}$, i.e., $U \cap T \notin L(k')$. On the other hand

$$k \in \overline{U \cap T}$$

for otherwise there would be an open neighborhood $k \in W \subset K$ with $W \cap U \cap T = \varnothing$, but this contradicts that T is dense in K.

Let $S \in \mathrm{Top}$. The sets of cardinality $\leqslant 2^{2^{\sharp S}}$ up to isomorphism is a set. Moreover, the choices of all topologies on these sets forms a set. Finally, all continuous maps from S to these topological spaces forms a set. We claim this is a solution set. Indeed, given a map $f : S \to K$ ($K \in \mathrm{CptHaus}$), we may replace K by $\overline{f(S)}$, which is again in CptHaus and assume $T := f(S)$ is dense in K. Then $\sharp K \leqslant 2^{2^{\sharp T}} \leqslant 2^{2^{\sharp S}}$ and f factors in the claimed way. □

5.5 The adjoint functor theorem for presentable categories

In this section, we briefly discuss – without proof – the adjoint functor theorem for a wide class of categories called presentable categories. This theorem is due to Adamek and Rosicky [**AdamekRosicky:Locally**]. They call these categories locally presentable, but we will follow the terminology of [**Lurie:HTT**].

Throughout, let κ be a *regular cardinal*, i.e., κ is infinite and cannot be expressed as $\kappa = \sum_{i<\alpha} \kappa_i$ with $\kappa_i < \kappa$ and $\alpha < \kappa$. For example $\aleph_0, \aleph_1, \dots$ are regular cardinals, but $\aleph_\omega = \sum_{i<\omega} \aleph_i$ is not regular.

Definition 5.29.

- A category J is called κ-small if it has fewer than κ morphisms.
- A category C is called κ-*filtered* if every diagram $K : J \to C$ with J being κ-small admits a cocone in C.
- Suppose C has all κ-filtered colimits. An object $c \in C$ is called κ-*compact* (or κ-*presentable*) if

  $$\mathrm{Hom}_C(c, -) : C \to \mathrm{Set}$$

  preserves κ-filtered colimits.

Example 5.30. (1) An $\aleph_0$-small category is a finite category (Definition 4.36), and thus $\aleph_0$-compact objects are just the compact objects encountered before.

(2) Suppose $\kappa < \kappa'$ and J, C and c as in the definition. Then we have the following implications:

$$J \text{ is } \kappa\text{-small} \Rightarrow J \text{ is } \kappa'\text{-small},$$
$$C \text{ is } \kappa\text{-filtered} \Leftarrow C \text{ is } \kappa'\text{-filtered},$$
$$c \text{ is } \kappa\text{-compact} \Rightarrow c \text{ is } \kappa'\text{-compact}.$$

(3) An object $X \in \mathrm{Set}$ is κ-compact iff its cardinality is less than κ.

(4) An object $X \in \mathrm{Top}$ is κ-compact iff it has the discrete topology and has cardinality $< \kappa$. This can be shown similarly to Exercise 4.11.

(5) The category $\mathrm{Ban}_{\leqslant 1}$ has all colimits, so we can consider κ-compactness. The one-dimensional space $\mathbf{C} \in \mathrm{Ban}_{\leqslant 1}$ is not ($\aleph_0$-)compact, but instead $\aleph_1$-compact. The key point is that $l_1 = \coprod_{n \in \mathbf{N}} \mathbf{C} = \mathrm{colim}(\mathbf{C} \to \mathbf{C}^2 \to \dots)$ is *not* an $\aleph_1$-filtered colimit, so that the computation in Example 4.43 does not apply. However, an *uncountable* coproduct $\coprod_{r \in R} V_r$ is a $\aleph_1$-filtered colimit (namely the filtered colimit indexed by the countable subsets $I \subset R$), and

$$\mathrm{Hom}_{\mathrm{Ban}_{\leqslant 1}}(\mathbf{C}, \coprod_{r \in R} V_r) = B_1(\coprod V_r) = \{v = (v_r), v_r \in V_r, \text{only countably many } v_r \neq 0, \sum |v_r| \leqslant 1\}.$$

Given such a (v_r), there is a countable subset $I \subset R$ such that $v \in \coprod_{r \in I} V_r (\subset \coprod_{r \in R} V_r)$, and therefore v factors over

$$\mathbf{C} \to \coprod_{r \in I} V_r.$$

This argument can be used to show that $\mathbf{C}$ is $\aleph_1$-compact. More generally, any countably-dimensional Banach space is $\aleph_1$-compact.

Definition 5.31. A category C is called *κ-presentable* if it has all (small) colimits and has a *set*(!) $A \subset \mathrm{Obj}(C)$ consisting of κ-presentable objects such that every object is a κ-filtered colimit of objects in A.
C is called *presentable* if it is κ-presentable for some regular cardinal κ.

Example 5.32. • The categories Set, Ab, Mon, Mod_R are all $\aleph_0$-presentable. Indeed, we can take A to be the set of finite sets of the form $\{1, \dots, n\}$ ($n \in \mathbf{N}$), respectively. Similarly, for Mod_R, say, there is a set of modules of the form

$$\mathrm{coker}(R^n \xrightarrow{f} R^m),$$

for $n, m \in \mathbf{N}$ and arbitrary maps f.

- The category of presheaves $\widehat{C}$ is $\aleph_0$-presentable for *any* category C.
- The category $\mathrm{Ban}_{\leqslant 1}$ is not $\aleph_0$-presentable but it is $\aleph_1$-presentable. An indication why this might hold is offered by Example 5.30, a full proof is in [**AdamekRosicky:Locally**].
- By (4), the category Top is not presentable.

The following theorem, due to Adamek and Rosicky can be deduced (in a non-trivial manner) from the adjoint functor theorem via the solution set condition.

Theorem 5.33. (*Adjoint functor theorem for presentable categories*) Let $F : C \to D$ be a functor between presentable categories. Then

- F is a right adjoint iff it preserves limits and κ-filtered colimits for *some* regular cardinal κ.
- F is a left adjoint iff it preserves colimits.

Example 5.34. This is a highly convenient adjoint functor theorem, it immediately produces the left adjoints of the functors

$$U : \mathrm{Grp} \to \mathrm{Set}$$
$$U : \mathrm{Mod}_S \to \mathrm{Mod}_S.$$

To pathological categories such as Top, it does however not apply, so the left adjoint to

$$U : \mathrm{CptHaus} \to \mathrm{Top}$$

is not ensured by this theorem, since Top is not presentable.

5.6 Exercises

Exercise 5.1. (Turning adjunctions into equivalences) Let $F : C \rightleftarrows D : G$ be an adjunction. Let $C' \subset C$ be the full subcategory consisting of the objects $X \in C$ such that the unit map $X \to G(F(X))$ is an isomorphism. Define $D' \subset D$ similarly. Show that F and G restrict to an equivalence of categories

$$F|_{C'} : C' \rightleftarrows D' : G|_{D'}.$$

Exercise 5.2. Let $F : C \rightleftarrows D : G$ be an adjunction. Show that the following are equivalent:
(1) F is fully faithful,
(2) the unit map $X \to G(F(X))$ is an isomorphism for each $X \in C$.

Exercise 5.3. A topological space X is *connected* if in any disjoint decomposition

$$X = V_1 \sqcup V_2$$

of *open* subsets, either $V_1 = \varnothing$ or $V_2 = \varnothing$. A topological space X is *locally connected* if it admits an open cover $X = \bigcup_{i \in I} U_i$ by connected spaces U_i.

- Show that X is locally connected iff there is a diagram

$$D : J \to \mathrm{Top}$$

  with $X = \operatorname{colim} D$ where the D_j are connected spaces.

- Show that the discrete topology functor D takes values in the full subcategory of $\mathrm{Top}^{\mathrm{loc.conn}} \subset \mathrm{Top}$ of locally connected spaces.

- Show that the functor $D : \mathrm{Set} \to \mathrm{Top}^{\mathrm{loc.conn}}$ admits a left adjoint π_0, given by taking the set of connected components of a space.

- Show that the space $\prod_{i \in \mathbf{N}} \{0, 1\}$ (each factor has the discrete topology, and the whole space has the product topology) is not locally connected. (*Note*: this space is a so-called profinite topological space, these are rarely locally connected.)

We end up with a quadruple of adjunctions

$$\begin{tikzcd}
\mathrm{Top}^{\mathrm{loc.conn.}} \arrow[r, bend left=50, "\pi_0"] \arrow[r, "U"] & \mathrm{Set} \arrow[l, bend right=20, "D"'] \arrow[l, bend left=30, "C"]
\end{tikzcd}$$

Exercise 5.4. (The universal property of the category of sets) Let C be a category admitting all small colimits. Let $\mathrm{Fun}'(\mathrm{Set}, C)$ be the full subcategory of $\mathrm{Fun}(\mathrm{Set}, C)$ consisting of the functors $F : \mathrm{Set} \to C$ that preserve all colimits. Show that the composite

$$\mathrm{Fun}'(\mathrm{Set}, C) \subset \mathrm{Fun}(\mathrm{Set}, C) \stackrel{F \mapsto F(\{*\})}{\longrightarrow} C$$

is an equivalence.

This statement is referred to as saying that Set is the *free cocompletion* of $\{*\}$ (the category with a single object and morphism). More generally, we will discuss in §8.3 that the Yoneda embedding (for a small category C)

$$C \subset \widehat{C}$$

exhibits as the free cocompletion of C.

Exercise 5.5. Let R be a commutative ring and M an R-module. Let $S \subset R$ be a submonoid with respect to the multiplication (also known as a multiplicatively closed nonempty subset).

Using solely the results from Exercise 4.19(1) and generalities about colimits, prove that:

(1) The localization of a flat R-module is flat. In particular, $R[S^{-1}]$ is a flat R-module.

(2) $M[S^{-1}] \otimes_R N = (M \otimes_R N)[S^{-1}]$.

Exercise 5.6. Give an example of a ring homomorphism $R \to S$ where the functors $S \otimes_R -$ (resp. $\mathrm{Hom}_R(S, -)$) don't preserve products (resp. coproducts), and give an example where they do.

Exercise 5.7. For a category C, the *arrow category* $\mathrm{Ar}(C)$ is defined as $\mathrm{Fun}(\{0 \to 1\}, C)$. I.e., objects in $\mathrm{Ar}(C)$ are just morphisms in C (referred to here as arrows for distinction), and morphisms between arrows are commutative squares.

Let $\Delta : C \to \mathrm{Ar}(C)$ be the diagonal functor, i.e. $X \mapsto (X \stackrel{\mathrm{id}}{\to} X) \in \mathrm{Ar}(C)$. Assume that C has pullbacks, pushouts and a *zero object*, i.e., an object that is at the same time a final object and an initial object. (For example, $C = \mathrm{Ab}$).

Show[1] that there is a chain of seven adjoints:

$$? \dashv ? \dashv ? \dashv \Delta \dashv ? \dashv ? \dashv ? \, .$$

Hint: the rightmost and leftmost adjoints are given by sending an arrow $Y \to Y'$ to the *kernel*, i.e., the pullback $0 \times_{Y'} Y$ and the *cokernel*, i.e., the pushout $Y \sqcup_{Y'} 0$.

Exercise 5.8. Let $R \to S$ be a ring homomorphism. Adapt the proof of Corollary 5.24 to show that the forgetful functor

$$\mathrm{Mod}_S \to \mathrm{Mod}_R$$

admits a left adjoint, denoted as $- \otimes_R S$.

Note: the most general tensor product functor is

$${}_A\mathrm{Mod}_B \times {}_B\mathrm{Mod}_C \stackrel{-\otimes_B -}{\to} {}_A\mathrm{Mod}_C.$$

Here ${}_A\mathrm{Mod}_B$ denotes the category of A-B-*bimodules*, i.e., abelian groups on which A acts from the left and B from the right and these two actions commute. The statement in Corollary 5.24 corresponds to $A = C = \mathbf{Z}$. The statement above is also a special case, using that any (associative) ring S is an S-S-bimodule. This functor can again be constructed using the adjoint functor theorem.

Exercise 5.9. Compute $(\mathbf{Z}/n) \otimes_{\mathbf{Z}} (\mathbf{Z}/m)$ by solely applying general facts about adjoints to the adjunction in Example 5.6 (which was verified in Exercise 5.8).

Exercise 5.10. (Unicity of adjoints) Let $L \dashv R$ be an adjunction. Suppose that there is another functor $L' : C \to D$ such that $L' \dashv R$ is also an adjunction. Show that there is a functorial isomorphism $L \cong L'$.

Hint: one can construct a transformation $L \Rightarrow L'$ by taking two appropriate transposes of id_C. Alternatively, one can argue by showing that there is a fully faithful functor

$$\Phi : \mathrm{Fun}(C, D) \to \mathrm{Fun}(C^{\mathrm{op}} \times D, \mathrm{Set}), L \mapsto \Phi(L) : ((c, d) \mapsto \mathrm{Hom}_D(L(c), d)).$$

(Functors $C^{\mathrm{op}} \times D \to \mathrm{Set}$ are called *profunctors*.) Then use that $\Phi(L) \cong \Phi(L')$, since both are isomorphic to the profunctor $\mathrm{Hom}_C(-, R(-))$.

Exercise 5.11. Prove Lemma 5.17.

Hint: in Lemma 4.26 we showed the functoriality of limits. The limit functor constructed in this way is a right adjoint (namely to the diagonal functor $\Delta : C \to \mathrm{Fun}(J, C)$). Dualize that proof and replace the functor category by an arbitrary category.

Exercise 5.12. Let

$$P_{\mathrm{conv}}(\mathbf{R}^n) \subset P(\mathbf{R}^n)$$

be the set of all *convex subsets* of $\mathbf{R}^n$. Regard both sets as partially ordered by inclusion and therefore as categories (Example 2.9). Use the adjoint functor theorem to construct the *convex hull* as the left adjoint of the above inclusion.

Exercise 5.13. Equip $\mathbf{Z}$ and $\mathbf{R}$ with their usual total order $\leqslant$. Describe the right and the left adjoint of the inclusion $\mathbf{Z} \subset \mathbf{R}$ (regarded as a functor between the categories associated to these ordered sets, cf. Example 2.9).

Exercise 5.14. Let $F \colon C \to D$ be a functor. Show that F admits a right adjoint if and only if the functor $\mathrm{Hom}_D(F(-), Y) \colon C^{\mathrm{op}} \to \mathrm{Set}$ is representable for every $Y \in D$.

Exercise 5.15. Let $f \colon X \to Y$ be a map between sets. Use Example 5.4 to explain why the preimage functor $f^{-1} \colon P(Y) \to P(X); U \mapsto f^{-1}(U)$ commutes with unions and intersections and the image functor $f \colon P(X) \to P(Y); U \mapsto f(U)$ only commutes with unions but not with intersections.

[1] This exercise is suggested by Ben Wieland, https://mathoverflow.net/q/46938.

Exercise 5.16. Let D be a complete poset category and $R\colon D \to C$ be a limit preserving functor into a locally small category C. Show that R admits a left adjoint.

Exercise 5.17. Show that the inclusion

$$\mathrm{FinSet} \to \mathrm{Set}$$

(of the category of finite sets into the category of all sets) does not admit a left adjoint nor a right adjoint.

Exercise 5.18. Let C be a category with all (small) coproducts and $F : C \to \mathrm{Set}$ a functor. Prove:
(1) if F has a left adjoint, then it is representable.
(2) if C has coproducts the converse holds: if F is representable then it has a left adjoint.
(3) Show by an example that one cannot drop the condition that C has coproducts in the second implication.

Exercise 5.19. The triangle identities also arise in so-called *symmetric monoidal categories*. This exercise explores them for the category Mod_R, where R is a commutative ring.

An R-module M is called *dualizable* iff there is another R-module M' and two maps (in Mod_R)

$$\eta : R \to M \otimes_R M', \epsilon : M' \otimes_R M \to R$$

(η is called the *unit map* (or *coevaluation*), ϵ is called the *counit map* or *evaluation*) such that the following composites are the respective identities:

$$M' = M' \otimes_R R \overset{\mathrm{id}_{M'} \otimes \eta}{\to} M' \otimes_R M \otimes_R M' \overset{\epsilon \otimes \mathrm{id}_{M'}}{\to} R \otimes_R M' = M',$$

$$M = R \otimes_R M \overset{\eta \otimes \mathrm{id}_M}{\to} M \otimes_R M' \otimes_R M \overset{\mathrm{id}_M \otimes \epsilon}{\to} M \otimes_R R = M.$$

(At the end, we use the canonical isomorphisms $M \otimes_R R \cong M$.) Prove the following:

- Suppose $R = k$ is a field. Show that a k-vector space V is dualizable iff it is finite-dimensional. This provides a basis-free characterization of finite-dimensionality.
- If M is dualizable, one may choose M' to be $M^* := \mathrm{Hom}_R(M, R)$ in the above definition.
- If M is dualizable, and $(\tilde{M'}, \tilde{\eta}, \tilde{\epsilon})$ is another datum as above, show that there is a unique isomorphism $M' \cong M''$ that is compatible with the unit and counit maps.
- (Voluntary) For a general commutative ring show that M is dualizable iff M is a finitely generated projective R-module.

  Hint: if M is dualizable, write $\eta(1) = \sum_{i=1}^n m_i \otimes m'_i \in M \otimes_R M'$ and use these elements to factor id_M as an appropriate map $M \to R^n \to M$. Deduce that M is finitely generated projective.

If M is dualizable and $f : M \to M$ is an R-module map, we define the trace to be the following map

$$\mathrm{tr}(f) : R \xrightarrow{\eta} M \otimes_R M' \overset{f \otimes \mathrm{id}_{M'}}{\to} M \otimes_R M' = M' \otimes_R M \xrightarrow{\epsilon} R.$$

- Prove that this is well-defined, i.e., independent of the choice of (M', η, ϵ).
- Let $f : V \to V$ be an endomorphism of a finite-dimensional vector space. Let $\mathrm{tr}'(f)$be the trace of f as defined in linear algebra. Show $\mathrm{tr}(f)$ is the map $k \to k$ given by multiplication with $\mathrm{tr}'(f)$.

Chapter 6

Abelian categories

In comparison to the wilderness of arbitrary categories, abelian categories are particularly well-behaved. They also form the technical backbone of a number of disciplines including commutative algebra, and homological algebra. This chapter is a brief invitation to this topic.

6.1 Definitions and examples

Recall from Exercise 4.8 that a *zero object*, usually denoted 0, in a category C is an object that is at the same time initial and terminal. If C has a zero object, it is also called a *pointed* category. A pointed category is called *semiadditive* if it admits finite coproducts and finite products and if the natural map

$$X \sqcup Y \to X \times Y$$

(from the coproduct to the product) of two arbitrary objects $X, Y \in C$, is an isomorphism. In this case either of the two isomorphic objects is also called a *biproduct* and is often denoted

$$X \oplus Y.$$

For a semiadditive category, the set $\mathrm{Hom}_C(X, Y)$ carries the structure of an abelian monoid:

- The zero morphism $0 \in \mathrm{Hom}_C(X, Y)$ is the unique morphism $X \to 0 \to Y$, where 0 is a zero object. (One checks (!) that for another zero object $0' \in C$, the composite $X \to 0' \to Y$ agrees with the previous one.)
- The sum $f + g$ of two morphisms $f, g \in \mathrm{Hom}_C(X, Y)$ is defined as the composite

  $$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xleftarrow{\cong} Y \sqcup Y \xrightarrow{\nabla} Y,$$

  where Δ is the diagonal, the map in the middle is the above isomorphism and ∇ is the so-called *codiagonal*, i.e., the unique map whose restriction to both copies of Y is id_Y. Again, one checks (!) that this is independent of the choice of the products and coproducts $X \times X$ etc.
- One checks the unitality, associativity and commutativity of the addition. For example, the commutativity $f + g = g + f$ follows from the following commutative diagram

  $$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, equal] & X \times X \arrow[r, "f \times g"] \arrow[d, "\sigma"] & Y \times Y \arrow[d, "\sigma"] & Y \sqcup Y \arrow[l, "\cong"'] \arrow[r, "\nabla"] \arrow[d, "\sigma"] & Y \arrow[d, equal] \\ X \arrow[r, "\Delta"] & X \times X \arrow[r, "g \times f"] & Y \times Y & Y \sqcup Y \arrow[l, "\cong"'] \arrow[r, "\nabla"] & Y. \end{tikzcd}$$

  Here σ denotes the maps that switch the two components.

Example 6.1.

- The category $\mathrm{Set}_* := \{*\}/\mathrm{Set}$ of pointed sets has a zero object, namely $\{*\} \xrightarrow{\mathrm{id}} \{*\}$. It is not semiadditive since the coproduct $X \sqcup Y$ of two pointed sets is the pushout $X \sqcup_{\{*\}} Y$ (with the two maps being the pointings of X and Y), while the product is the regular product in Set, $X \times Y$, pointed in the natural way. The natural map $X \sqcup_{\{*\}} Y \to X \times Y$ is not an isomorphism.
- The category AbMon of *abelian monoids* (and monoid homomorphisms) is semi-additive since the set-theoretic product $X \times Y$ of two abelian monoids, equipped with the usual monoid structure, is both a coproduct and a product.
- The category Mon of *monoids* (and monoid homomorphisms) is again pointed by the trivial monoid 0, but not semi-additive: the natural map

$$\mathbf{N} \sqcup \mathbf{N} \to \mathbf{N} \times \mathbf{N}$$

(coproduct, resp. product in Mon) is not an isomorphism. To see this, it suffices by the Yoneda lemma (Lemma 3.6) to see that mapping out of these objects gives non-isomorphic sets. Indeed

$$\mathrm{Hom}_{\mathrm{Mon}}(\mathbf{N} \sqcup \mathbf{N}, M) = M \times M,$$

while

$$\mathrm{Hom}_{\mathrm{Mon}}(\mathbf{N} \times \mathbf{N}, M) = \{(m', m'') \in M \times M, m'm'' = m''m'\}.$$

- Similarly, Grp is not semiadditive, but Ab and more generally Mod_R is semiadditive for any ring R.
- For any small category C and any abelian category A, the functor category $\mathrm{Fun}(C, A)$ is again abelian. Indeed, limits and colimits are computed pointwise.

Definition 6.2. A semiadditive category C is called *additive* if the abelian monoid $\mathrm{Hom}_C(X, Y)$ constructed above is in fact an abelian group.

Definition 6.3. An additive category C is called *abelian* if every morphism $f : X \to Y$ in C has a kernel and a cokernel (Exercise 4.27) and if the dotted natural map (which is the unique map making the square commute) is an isomorphism:

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "p"] & \operatorname{coker} f \\
& \operatorname{coker} i \arrow[r, dotted] & \ker p \arrow[u] &
\end{tikzcd} \tag{6.4}$$

Example 6.5.

- The category AbMon is semiadditive, but not additive and therefore not abelian.
- The category Grp of groups is not semiadditive. It does have kernels and cokernels, so the condition in Definition 7.3 makes sense. However, this condition is not satisfied: for a group homomorphism $f : X \to Y$, the kernel $\ker f = \{x \in X, f(x) = e_Y\}$ is the usual kernel. The cokernel $\operatorname{coker} f = Y/N_Y(f(X))$ is the quotient of Y by the normalizer of the subgroup $f(X) \subset Y$ (!) . In particular, if f is injective, the above diagram simplifies to

$$\begin{tikzcd}
\{e_X\} \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "p"] & Y/N_Y(X) \\
& X \arrow[r, dotted] & N_Y(X) \arrow[u] &
\end{tikzcd}$$

and the bottom map is an isomorphism iff $X \subset Y$ is a normal subgroup.

- The category Ab and more generally Mod_R is abelian.
- The category of Hausdorff topological abelian groups is not abelian (Exercise 7.4).

- If a category C is abelian, then C^{op} is also abelian. Indeed, the axioms in the definition of an abelian category are symmetric. A non-trivial fact in harmonic analysis, *Pontryagin duality*, establishes an equivalence
$$\mathrm{Hom}(-, S^1) : \mathrm{Ab}^{\mathrm{op}} \cong \mathrm{CompAb}$$
between $\mathrm{Ab}^{\mathrm{op}}$ and the category of compact Hausdorff spaces, that are also abelian groups.

The following nontrivial theorem due to Freyd shows that the last examples are prototypical. To state it, we need another definition:

Definition 6.6. A functor $C \to D$ is *left exact* (resp. *right exact*, resp. *exact*) if it preserves finite limits (resp. finite colimits, resp. finite limits and finite colimits).

Example 6.7. Any left adjoint is right exact, since it preserves all colimits. Dually, any right adjoint is left exact. The property when, in addition, a given left adjoint is also left exact, leads to interesting notions.

For example, for a commutative ring R and any R-module M, the functor
$$M \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_R$$
is right exact. The module M is called *flat* iff it is left exact (cf. Definition and Lemma 6.17 for some discussion).

The functors
$$\mathrm{Hom}_R(M, -) : \mathrm{Mod}_R \to \mathrm{Mod}_R, \text{respectively } \mathrm{Hom}_R(-, M) : \mathrm{Mod}_R^{\mathrm{op}} \to \mathrm{Mod}_R$$
are left exact. The module M is called *projective* (resp. *injective*) if the functors are exact.

Theorem 6.8. (*Freyd-Mitchell embedding theorem*) If C is a small abelian category, there is a (possibly non-commutative) ring R and a fully faithful, exact functor
$$C \to \mathrm{Mod}_R.$$

6.2 Elementary properties of abelian categories

Abelian categories have very special properties. Here is a small selection of such facts:

Lemma 6.9. Let C be an abelian category. Then any morphism $f : X \to Y$ has a factorization
$$f = m \circ e$$
with m a monomorphism and e an epimorphism. An example of such a factorization is
$$m = \ker \operatorname{coker} f, \ e = \operatorname{coker} \ker f.$$
This factorization has the property that for another map f' as in the following commutative diagram, there is a unique morphism k making the whole diagram commutative:
$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "g"] & Z \arrow[r, "m"] \arrow[d, dotted, "k"] & Y \arrow[d, "h"] \\ X' \arrow[r, "e'"] & Z' \arrow[r, "m'"] & Y'. \end{tikzcd}$$

Proof. The existence of such a factorization follows from the definition, cf. (7.4). To show the existence and unicity of k, consider
$$\begin{tikzcd} \ker f \arrow[r, "u"] & X \arrow[r, "e"] \arrow[d, "g"] & Z \arrow[r, "m"] \arrow[d, dotted, "k"] & Y \arrow[d, "h"] \\ & X' \arrow[r, "e'"] & Z' \arrow[r, "m'"] & Y'. \end{tikzcd}$$

We claim $\ker f = \ker e$. Indeed, we have

$$\mathrm{Hom}_C(T, \ker f) = \{t : T \to X, f \circ t (= m \circ (e \circ t)) = 0\}.$$

Since m is a monomorphism this is equivalent to $e \circ t = 0$, i.e., we get $\mathrm{Hom}_C(T, \ker e)$, so the claim follows from the Yoneda lemma.

We have $0 = hfu = m'e'gu$, so that $e'gu = 0$ (m' monic) and $e'g$ factors over $e = \operatorname{coker} u$ (using that C is abelian), as $e'g = ke$ for a unique map k as displayed. Then $m'ke = m'e'g = hme$ so that $m'k = hm$, finishing the proof. □

Lemma 6.10. Let $f : A \to B$ be a morphism in an abelian category. Then the following are equivalent:

(1) f is a *pseudo-epimorphism*, i.e., for any $g : B \to C$ with $gf = 0$ we have $f = 0$,

(2) f is an epimorphism,

(3) f is a *normal epimorphism*, i.e., the cokernel of some map,

(4) f is the cokernel of its kernel, $f = \operatorname{coker} \ker f$.

Proof. The implications (4) ⇒ (3) ⇒ (2) ⇒ (1) are trivial (and hold in any pointed category).

(1) ⇒ (2): If two maps $g, h : B \to C$ satisfy $gf = hf$ then $(g-h)f = 0$ by Exercise 7.1, so that $g-h = 0$ and thus $g = h$.

(2) ⇒ (3): we have a factorzation $f = m \circ e$ with $m = \ker \operatorname{coker} f$ and $e = \operatorname{coker} \ker f$. Since f is an epimorphism $f = \mathrm{id}_B \circ f$ is another such factorization. By the unicity of the factorization (Lemma 7.9), we see that f is isomorphic to m, and thus an epimorphism as well.

(3) ⇒ (4): Suppose f is the cokernel of a map g, as displayed:

$$\begin{tikzcd}
\bullet \arrow[dr, "q:=\ker f"] & & B \arrow[dd, shift left, "m"] \\
& A \arrow[ur, "f"'] \arrow[dr, "k:=\operatorname{coker} q"'] & \\
\bullet \arrow[uu, "h"] \arrow[ur, "g"] & & \bullet \arrow[uu, shift left, "l"]
\end{tikzcd}$$

There are maps h and l (making everything commutative) as indicated since $fg = 0$ and $fq = 0$. We have $kg = kqh = 0$, so that there is also a map m with $mf = k$. To check $ml = \mathrm{id}$ and $lm = \mathrm{id}$ we may precompose with the epimorphisms f and k, respectively, which then holds by the construction of the maps. □

Lemma 6.11. Let

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\
C \arrow[r, "k"] & D
\end{tikzcd}$$

be a pullback diagram (i.e., the natural map $A \to B \times_D C$ is an isomorphism) in an abelian category with h an epimorphism. Then

(1) the square is a pushout square (i.e., the natural map $B \sqcup_A C \to D$ is an isomorphism),

(2) g is an epimorphism.

Remark 6.12. The second point asserts that epimorphisms are stable under pullback in an abelian category. Note that by (the dual of Exercise 4.23), epimorphisms are stable under pushouts in *any* category.

Proof. Before embarking on the proof proper, we observe the following:

(1) the square is commutative iff the composite $A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \oplus C \xrightarrow{(h,-k)} D$ is zero. Indeed, that composite is just $hf - kg$. Here $\begin{pmatrix} f \\ g \end{pmatrix} : A \to B \oplus C (= B \times C)$ etc.

(2) the square is a pullback iff $\begin{pmatrix} f \\ g \end{pmatrix} = \ker(h, -k)$. Indeed, the latter kernel is just the pullback $B \times_D C$: $\mathrm{Hom}(T, \ker(h, -k))$ is the set of maps $f : T \to B \oplus C$ such that the composites $T \xrightarrow{f} B \oplus C (= B \sqcup C) \to B \xrightarrow{h} D$ and $T \xrightarrow{f} B \oplus C \to C \xrightarrow{k} D$ agree. This agrees with $\mathrm{Hom}(T, B \times_D C)$.

(3) Dually, the square is a pushout iff $(h, -k) = \mathrm{coker} \begin{pmatrix} f \\ g \end{pmatrix}$, as one proves similarly (!) .

We now prove the statement proper. By the second point, we have $\begin{pmatrix} f \\ g \end{pmatrix} = \ker(h, -k)$. The map h is an epimorphism by assumption, so that $(h, -k) : B \oplus C \to D$ is an epimorphism as well. By Lemma 7.10, it is the cokernel of its kernel, i.e., $(h, -k) = \mathrm{coker} \begin{pmatrix} f \\ g \end{pmatrix}$, so we do have a pushout.

In order to show g is an epimorphism it suffices to take a map $x : C \to E$ as pictured with $xg = 0$, and show that $x = 0$ (Lemma 7.10). If $xg = 0$ we have a commutative diagram like so:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \arrow[ddr, "0"] & \\
C \arrow[r, "k"] \arrow[drr, "x"'] & D \arrow[dr, dotted, "y"] & \\
& & E
\end{tikzcd}$$

By the universal property of the pushout, we get a unique map y with $yh = 0$. Since h is an epimorphism this gives $y = 0$, so that $x = yk = 0$. $\square$

6.3 Exercises

Exercise 6.1. Suppose C is semi-additive. Let $X \in C$ and $Y \xrightarrow{f} Z$ be a morphism. Show that the maps

$$\mathrm{Hom}_C(X, Y) \xrightarrow{f \circ -} \mathrm{Hom}_C(X, Z),$$

$$\mathrm{Hom}_C(Z, X) \xrightarrow{- \circ f} \mathrm{Hom}_C(Y, X)$$

are monoid homomorphisms.

Thus, if C is additive, these maps are homomorphisms of abelian groups. One refers to this situation by saying that an additive category is *enriched over abelian groups*: its Hom-sets are in fact abelian groups and the composition maps

$$\mathrm{Hom}_C(Y, Z) \times \mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_C(X, Z)$$

are bilinear.

Exercise 6.2. Prove:

(1) Any isomorphism (in any category) is an epimorphism and a monomorphism.

(2) The converse holds in an *abelian* category.

(3) Show that (2) also holds in Set, but not in CRing (consider $\mathbf{Z} \to \mathbf{Q}$).

Exercise 6.3. State formally (and prove) the following assertion: the factorization of a morphism f in an abelian category into an epimorphism followed by a monomorphism is functorial.

Exercise 6.4. Let C be the category of Hausdorff topological abelian groups: its objects are topological Hausdorff spaces X equipped with a *continuous* map $+ : X \times X \to X$ satisfying the usual conditions on an abelian group); its morphisms are maps that are both continuous and group homomorphisms.

(1) Show that the category is additive.

(2) Show that the kernel of a map $f : X \to Y$ is the usual kernel, equipped with the subspace topology.

(3) Show that the cokernel of $f : X \to Y$ is given by $Y/\overline{f(X)}$, the quotient by the (topological) closure of the image $f(X)$.

(4) Let G be a non-discrete topological group (such as $G = \mathbf{R}$). Apply Exercise 7.2 to the map $\mathrm{id} : G' \to G$, where G' is G, but equipped with the discrete topology, in order to show that the category C is not abelian.

Chapter 7

Monads

Monads are endofunctors

$$T : C \to C$$

equipped with a multiplication and unit map, familiar from the case of monoids. A typical example is the functor

$$T : \mathrm{Set} \to \mathrm{Set}, X \mapsto R[X],$$

sending a set to the (set underlying the) free R-module (for a fixed ring R) on the set X. In this case the multiplication of R gives rise to a "multiplication" $T \circ T \to T$, and the unit $1 \in R$ gives rise to the "unit" $\mathrm{id} \to T$. Just as every ring R carries with it its category of modules Mod_R, every monad comes along with its category of algebras, Alg_T. (The name algebras is standard, but for the above monad, $\mathrm{Alg}_{R[-]}$ is exactly the category of R-modules.)

Monads are closely related to adjunctions, by virtue of the following dictionary (U stands for a forgetful functor)

$$\begin{tikzcd} \text{adjunctions} \arrow[r, bend left, "L \dashv R \mapsto T := RL"] & \text{monads} \arrow[l, hook, bend left, "T \mapsto \mathrm{Free} \dashv U"] \end{tikzcd} .$$

As soon as these notions are in place, one immediately checks that the composition $T \mapsto \mathrm{Free} \dashv U \mapsto U \circ \mathrm{Free}$ is the identity. (This is indicated by the inclusion sign at the lower arrow.) Conversely, it is *not* always true that for an adjunction $L \dashv R$, the $\mathrm{Free} \dashv U$-adjunction for the monad $T := RL$ is the original adjunction. Those adjunction where this *is* correct are called *monadic adjunctions*. Understanding that condition and identifying criteria, known as *monadicity theorems*, when that happens are a crucial topic in category theory. For example, this helps tell apart the adjunctions

$$D : \mathrm{Set} \rightleftarrows \mathrm{Top} : U$$

from

$$\beta : \mathrm{Set} \rightleftarrows \mathrm{CptHaus} : U.$$

Another classical application of monadicity for (opposite) categories arises in algebraic geometry by the name of faithfully flat descent.

7.1 Definitions and examples

Definition 7.1. Let C be a category. A *monad* on C is a triple (T, η, μ) consisting of an endofunctor

$$T : C \to C,$$

a natural transformation (called the *unit* of the monad)

$$\eta : \mathrm{id}_C \Rightarrow T,$$

and a natural transformation (called *multiplication*)

$$\mu : T \circ T \Rightarrow T$$

(between functors from C to C), such that the following diagrams are commutative:

$$\begin{tikzcd} T \circ T \circ T \arrow[r, "T(\mu)"] \arrow[d, "\mu T"'] & T \circ T \arrow[d, "\mu"] \\ T \circ T \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[d, "T\eta"] \arrow[dr, equal] & T \circ T \arrow[d, "\mu"] \\ T \circ T \arrow[r, "\mu"] & T. \end{tikzcd}$$

Example 7.2. Let R be a ring. We define a monad on the category of sets as follows: $T : \mathrm{Set} \to \mathrm{Set}$ sends a set X to the free R-module generated by X regarded as a set (i.e., we forget that it is an R-module), i.e.,

$$T(X) = \bigoplus_{x \in X} R.$$

For $x \in X$, we will consistently denote by $e_x = (\dots, 0, 1, 0, \dots)$ the basis vector with the 1 at the spot corresponding to x. For a map $f : X \to Y$, $T(X) \to T(Y)$ is then the map sending e_x to $e_{f(x)}$ (and R-linearly extending; note though that $T(f)$ is just a map of sets). This is a monad: the unit map $X \to T(X)$ sends $x \in X$ to e_x To define the multiplication map, it is useful to notice that

$$T(X) = U\,\mathrm{Free}(X),$$

where

$$\mathrm{Free} : \mathrm{Set} \rightleftarrows \mathrm{Mod}_R : U$$

is the free-forgetful adjunction. Now, the map

$$TT(X) = U\,\mathrm{Free}(U\,\mathrm{Free}(X)) \to T(X) = U\,\mathrm{Free}(X)$$

is defined to be the underlying map of sets associated to the map

$$\mathrm{Free}(U\,\mathrm{Free}(X)) \to \mathrm{Free}(X)$$

of R-modules, which reads

$$\bigoplus_{t \in U\,\mathrm{Free}(X)} R \to \mathrm{Free}(X).$$

Let $t = \sum r_x x$ be an element of $U\,\mathrm{Free}(X)$. Then the multiplication map sends $e_t (\in \mathrm{Free}(U\,\mathrm{Free}(X))$ to $t \in \mathrm{Free}(X) = \bigoplus_{s \in X} R$. This shows that the structure of the adjunction gives rise to the monad data. We will revisit this for an abstract adjunction and verify the unitality and associativity condition for such more general mondas below.

Lemma 7.3. Let $L : C \rightleftarrows D : R$ be an adjunction. Then the functor

$$T := R \circ L : C \to C$$

is a monad, whose unit $\eta : \mathrm{id} \to T$ is the unit of the adjunction and whose multiplication is (essentially) given by the counit:

$$\mu : T \circ T = RLRL \overset{R\epsilon L}{\to} RL = T.$$

Proof. This is a consequence of the triangle identities. For example,

$$\begin{tikzcd} RL \arrow[r, "\eta RL"] \arrow[d, "RL\eta"'] \arrow[dr, equal] & RLRL \arrow[d, "\mu"] \\ RLRL \arrow[r, "R\epsilon L"] & RL, \end{tikzcd}$$

so the commutativity of the lower left triangle is the triangle identity $\epsilon L \circ L\eta = \mathrm{id}_L$ (and applying R).

The associativity follows from naturality of the counit map as follows: since the counit ϵ is natural, we get for any map $f : X \to Y$ in D a commutative diagram

$$\begin{tikzcd} LRX \arrow[r, "LRf"] \arrow[d, "\epsilon_X"'] & LRY \arrow[d, "\epsilon_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Applying this to $f = \epsilon_Y : LRY \to Y$, this reads

$$\begin{tikzcd} LR(LR)Y \arrow[r, "LR\epsilon_Y"] \arrow[d, "\epsilon_{LRY}"'] & LRY \arrow[d, "\epsilon_Y"] \\ LRY \arrow[r, "\epsilon_Y"] & Y. \end{tikzcd}$$

Picking finally $Y = LA$ for some $A \in C$ and also applying R, we obtain the desired commutative square expressing the associativity condition:

$$\begin{tikzcd} RLR(LR)(LA) \arrow[r, "RLR\epsilon_{LA}"] \arrow[d, "R\epsilon_{LRLA}"'] & RLR(LA) \arrow[d, "R\epsilon_{LA}"] \\ RLR(LA) \arrow[r, "R\epsilon_{LA}"] & RLA. \end{tikzcd} \tag{7.4}$$

Example 7.5. Let R be a commutative ring. The adjunction

$$\mathrm{Hom}_R(-, R) : \mathrm{Mod}_R^{\mathrm{op}} \rightleftarrows \mathrm{Mod}_R : \mathrm{Hom}_R(-, R)$$

gives rise to a monad on Mod_R, known as the *bidualization monad*:

$$\mathrm{Mod}_R \to \mathrm{Mod}_R, M \mapsto M^{**}(:= \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)).$$

The unit map is the usual map

$$M \to M^{**}, m \mapsto (f \mapsto f(m)).$$

Example 7.6. The power set functor

$$P : \mathrm{Set} \to \mathrm{Set}$$

defined by $X \mapsto P(X)$, and $f : X \to Y$ to the map $P(X) \to P(Y)$, $(X \supset)U \mapsto f(U)(\subset Y)$. This functor is a monad, called the *power set monad*: the unit map is given by

$$\eta_X : X \to P(X), x \mapsto \{x\}.$$

The multiplication map

$$\mu_X : P(P(X)) \to P(X)$$

sends a subset of $A \subset P(X)$, i.e., a set $U_a \subset X$ (indexed by the set A) of subsets of X, to $\bigcup_{a \in A} U_a$. These are indeed natural transformations, e.g., for $f : X \to Y$, this expresses

$$\bigcup_{a \in A} f(U_a) = f\left(\bigcup_{a \in A} U_a\right).$$

The conditions on a monad can be painfully verified by hand or alternatively by using the fact that there is an adjunction

$$P(-) : \mathrm{Set} \rightleftarrows \text{complete sup-lattices} : U.$$

Here at the right we have the category of posets having suprema (i.e., colimits!), and suprema-preserving (i.e., colimit-preserving) functors.

7.2 Algebras over a monad

Definition 7.7. Let $T : C \to C$ be a monad. The category of *T-algebras* (a.k.a. the *Eilenberg-Moore category*) Alg_T is the following. Objects are pairs (A, α) with $A \in C$ and $\alpha : T(A) \to A$ a morphism in C such that the two diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & T(A) \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T(T(A)) \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"] & A. \end{tikzcd}$$

Morphisms in Alg_T are morphisms $f : A \to A'$ such that the following square commutes:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"] & TA' \arrow[d, "\alpha'"] \\ A \arrow[r, "f"] & A' \end{tikzcd} .$$

These conditions look familiar from the definition of an R-module M. Indeed, this is no coincidende:

Example 7.8. Let $T : \mathrm{Set} \to \mathrm{Set}$ be the monad associated to the free-forgetful adjunction associated to a ring R, as above. We unwind the notion of a T-algebra. Let (A, α) be such an algebra, i.e., $A \in \mathrm{Set}$. Given the map $\alpha : T(A) = \bigoplus_{x \in A} R \to A$ we define the scalar multiplication by R as the map

$$R \times A \to A, (r, x) \mapsto \alpha(re_x),$$

where $e_x \in T(A)$ is the basis vector for $x \in A$. The diagram

$$\begin{tikzcd} A \arrow[r, "x \mapsto e_x"] \arrow[dr, equal] & \bigoplus_{x \in A} R \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

then means $1_R \cdot x = x$. The commutativity of

$$\begin{tikzcd} T(T(A)) \arrow[r, "T\alpha"] \arrow[d, "\mu_A"] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"] & A. \end{tikzcd}$$

is equivalent to the associtavity of the R-multiplication on A (!) . We also define the addition on A as

$$A \times A \to A, (x, y) \mapsto \alpha(e_x + e_y).$$

It is a further routine check to verify the remaining conditions for being an R-module, such as distributivity etc.

Definition and Lemma 7.9. Let $T : C \to C$ be a monad. Then the forgetful functor

$$U : \mathrm{Alg}_T(C) \to C, (X, \alpha) \mapsto X$$

admits a left adjoint, denoted Free, and called the *free T-algebra functor*. It sends $X \in C$ to

$$(TX, \mu_X : T(TX) \to TX).$$

A T-algebra of this form is called a *free T-algebra*. The full subcategory of Alg_T consisting of free T-algebras is called the *Kleisli category*.

Proof. It is routine to check the functoriality of Free. To show it is an adjunction observe that a map

$$g : (TX, \mu_X) \to (A, \alpha)$$

yields by composition with the map $\eta_X : X \to TX$ a map

$$X \xrightarrow{\eta_X} TX \xrightarrow{g} A.$$

Conversely, a map $f : X \to A$ gives rise to a map

$$TX \xrightarrow{Tf} TA \xrightarrow{\alpha} A.$$

One checks that this map is a map of T-algebras. One further checks that these two constructions are indeed inverse to each other: for example, given a map g as above, we need to check that $\alpha \circ Tg \circ T\eta_X$ is the original map g, which results from the commutativity of the following diagram.

$$\begin{tikzcd}
TX \arrow[r, "T\eta_X"] \arrow[dr, equal] & TTX \arrow[r, "Tg"] \arrow[d, "\mu"] & TA \arrow[d, "\alpha"] \\
& TX \arrow[r, "g"] & A.
\end{tikzcd}$$

The left triangle is the unitality condition for T and the right one is part of α being a morphism of T-algebras. □

7.3 Monadic adjunctions

So far, we have established a dictionary:

- a monad $T : C \to C$ gives rise to an adjunction

$$\mathrm{Free} : C \rightleftarrows \mathrm{Alg}_T(C) : U.$$

- an adjunction

$$L : C \rightleftarrows D : R \tag{7.10}$$

  gives rise to a monad

$$T := RL : C \to C.$$

If we start with a monad T, then the monad associated to the free-forgetful adjunction gives back the same monad. This follows directly from the definition of the free algebra functor.

In order to adress the converse question we use the following notion:

Definition and Lemma 7.11. Let $L \dashv R$ be an adjunction as in (6.10). Then R factors as

$$D \xrightarrow{\tilde{R}} \mathrm{Alg}_T(C) \xrightarrow{U} C,$$

where $\tilde{R}(d)$ is $R(d)$, equipped with the T-action given by

$$T(R(d)) = RLR(d) \xrightarrow{R\epsilon_d} R(d).$$

The adjunction is called a *monadic adjunction* if the functor $\tilde{R}$ is an equivalence of categories.

Proof. One checks that the map $R\epsilon_d$ indeed does define a T-algebra structure on $R(d)$, as in (6.4) (replace LA there by d)(!) . □

Example 7.12. • Let $R \to S$ be a ring homomorphism. The adjunction

$$S \otimes_R - : \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_S : Forget \tag{7.13}$$

is a monadic adjunction. Indeed, the monad $T : \mathrm{Mod}_R \to \mathrm{Mod}_R$ sends $M \mapsto S \otimes_R M$ (regarded as an R-module). A T-algebra is then an R-module M with a R-linear map

$$T(M) = S \otimes_R M \to M.$$

By the tensor-Hom-adjunction this is the same as an R-linear map $M \to \mathrm{Hom}_R(S, M)$, or, yet equivalently, an R-bilinear map $S \times M \to M$. This map can be used to define the S-action on M: $(s, m) \mapsto s \cdot m$. This indeed defines an S-module structure, as follows from inspecting the definition of a T-algebra.

- We have already seen above that the adjunction

$$R[-] : \mathrm{Set} \rightleftarrows \mathrm{Mod}_R : U$$

is monadic.

- By similar arguments, the adjunction

$$\mathrm{Free} : \mathrm{Set} \rightleftarrows \mathrm{Grp} : U$$

is also monadic, as we will see in Exercise 6.4.

- The discrete-forgetful adjunction

$$D : \mathrm{Set} \rightleftarrows \mathrm{Top} : U$$

(D puts the discrete topology on a set, U forgets the topology) is *not* monadic: the associated monad $T = UD$ is the identity functor on Set, so that the factorization in Definition and Lemma 6.11 reads

$$\mathrm{Top} \xrightarrow{\tilde{U}} \mathrm{Alg}_T(\mathrm{Set}) = \mathrm{Set} \xrightarrow{U} \mathrm{Set}.$$

In particular, $\tilde{U}$ fails to be an equivalence.

A more general, abstract necessary condition to be monadic is supplied by the following lemma. It again shows that the above adjunction is not monadic: R is not conservative (there are continuous bijective maps that fail to be homeomorphisms).

Lemma 7.14. For a monad T on a category C, the forgetful functor $U : \mathrm{Alg}_T(C) \to C$ is conservative.

Proof. Given a T-algebra map $f : A \to A'$ that is an isomorphism in C, say with inverse $g = U(f)^{-1} =: f^{-1}$, we claim that g is naturally a T-algebra map. Indeed, consider the following diagram (in C):

$$\begin{tikzcd} TA' \arrow[r, "\alpha'"] \arrow[d, "Tg"] & A' \arrow[d, "g"] \\ TA \arrow[r, "\alpha"] \arrow[d, "{Tf, \cong}"] & A \arrow[d, "{f, \cong}"] \\ TA' \arrow[r, "\alpha'"] & A'. \end{tikzcd}$$

The bottom square commutes since f is an algebra map. The two vertical composites are the identities since T is functorial. Thus the outer square commutes. The two bottom vertical maps are isomorphisms. Therefore the top square commutes. The unitality condition for g is checked similarly. □

Corollary 7.15. If $L \dashv R$ is a monadic adjunction, then R is conservative.

Proof. Indeed, both the equivalence $\tilde{R}$ and the forgetful functor $\mathrm{Alg}_T(C) \to C$ are conservative. □

7.4 Monadicity theorems

Monadicity theorems give sufficient (and, in some cases, necessary) conditions for an adjunction to be monadic. The following theorem is known as the "crude" monadicity theorem, due to Beck in the 1960's. Recall from Remark 4.46 that a *reflexive coequalizer* is a colimit of shape

$$\begin{tikzcd} 1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & 0 \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

(two objects, three non-identity morphisms and $f \circ \sigma = g \circ \sigma = \mathrm{id}_0$.) This is the full subcategory of $\boldsymbol{\Delta}$ (Example 4.56) consisting of the objects $[0]$ and $[1]$. Its non-full subcategory

$$1 \underset{g}{\overset{f}{\rightrightarrows}} 0$$

is a cofinal subcategory. In particular a reflexive coequalizer can be computed as the coequalizer of the diagram obtained by omitting the σ.

Theorem 7.16. (*Crude monadicity theorem*) Let $L : C \rightleftarrows D : R$ be an adjunction. Suppose
(1) D admits and R preserves reflexive coequalizers,

(2) R is conservative.
Then the adjunction is a monadic adjunction.

Before proving the theorem, we explore its reach by means of several examples. First, the theorem shows (again) that the tensor-forgetful adjunction is monadic (cf. Example 6.12): the forgetful functor is conservative and preserves all colimits by virtue of having a right adjoint, $\mathrm{Hom}_S(R, -)$, cf. Example 5.6.

Let $R \to S$ be a ring homomorphism of commutative rings. Passing to the opposite categories in (6.13), we obtain an adjunction

$$Forget^{\mathrm{op}} : (\mathrm{Mod}_S)^{\mathrm{op}} \rightleftarrows (\mathrm{Mod}_R)^{\mathrm{op}} : (S \otimes_R -)^{\mathrm{op}}.$$

Let us examine whether it is monadic (or, in common terminology, the adjunction (6.13) is *comonadic*). For any R, $\mathrm{Mod}_R^{\mathrm{op}}$ admits even all colimits, since Mod_R admits all limits.

Definition and Lemma 7.17. In the above situation, the following are equivalent. In this event, S is called a *flat* R-algebra.
(1) $S \otimes_R -$ preserves finite limits,

(2) $S \otimes_R -$ preserves equalizers,

(3) $S \otimes_R -$ preserves kernels, i.e., for any map $f : M \to N$ of R-modules, we have

$$\ker(S \otimes_R M \overset{\mathrm{id}_S \otimes f}{\longrightarrow} S \otimes_R N) = S \otimes_R \ker f.$$

(4) $S \otimes_R -$ preserves injective maps, i.e., if $f : M \to N$ is an injective map of R-modules, then $\mathrm{id}_S \otimes f : S \otimes_R M \to S \otimes_R N$ is also injective.

Proof. Clearly (1) $\Rightarrow$ (2). Conversely, being a left adjoint the functor always preserves coproducts, hence finite products (Mod_R is additive). Then use Proposition 4.32.

We have (2) $\Leftrightarrow$ (3) since an equalizer of a diagram $M \underset{g}{\overset{f}{\rightrightarrows}} N$ is also the equalizer of the diagram $M \underset{0}{\overset{f-g}{\rightrightarrows}} N$, i.e., the kernel of f.

For (3) $\Leftrightarrow$ (4) note that in Mod_R, a map f is a monomorphism iff it is injective. Then $\Rightarrow$ follows since f is injective iff $\ker f = 0 \overset{(3)}{\Rightarrow} \ker(\mathrm{id}_S \otimes f) = 0$ iff $\mathrm{id}_S \otimes f$ is injective. Conversely, any monomorphism is the kernel of a map (in fact, the kernel of its cokernel) by (the dualized version of) Lemma 7.10. $\square$

Definition and Lemma 7.18. Let S be a flat R-algebra. Then, the following are equivalent. In this event, S is called a *faithfully flat* R-algebra.
(1) $S \otimes_R -$ is conservative,

(2) For $M \in \mathrm{Mod}_R$, we have $S \otimes_R M = 0 \Rightarrow M = 0$.

Proof. Indeed, $M = 0$ iff $0 : M \to M$ is an isomorphism. Conversely, for $f : M \to N$, f is an isomorphism iff $\ker f$ and $\operatorname{coker} f$ are zero. By flatness, tensoring preserves the kernel (and always preserves colimits such as the cokernel). Thus $S \otimes_R (\ker f) = \ker(S \otimes f) = 0$ implies $\ker f = 0$ etc. □

Example 7.19.
- For any commutative ring R, the map $R \to R[t]$ is faithfully flat: $R[t] \otimes_R M = \bigoplus_{n \geqslant 0} M$ (as an R-module). This functor preserves injections and is conservative.
- The map $\mathbf{Z} \to \mathbf{Q}$ is flat (either by inspection or alternatively using Exercise 5.5), but not faithfully flat:
$$\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}/p = \mathbf{Q}/p = 0.$$
- The unique ring homomorphism $R[t] \to R$ satisfying $t \mapsto 0$ is not flat: we have
$$0 = \ker(R[t] \xrightarrow{\cdot t} R[t]),$$
and after applying $R \otimes_{R[t]} -$, we obtain the map
$$R \xrightarrow{\cdot 0} R,$$
whose kernel is
$$R = \ker(R \xrightarrow{0} R),$$
so that $R[t] \otimes_R -$ does not preserve kernels.

Corollary 7.20. Suppose S is faithfully flat. Then the adjunction
$$Forget^{\mathrm{op}} : (\mathrm{Mod}_S)^{\mathrm{op}} \rightleftarrows (\mathrm{Mod}_R)^{\mathrm{op}} : (S \otimes_R -)^{\mathrm{op}}$$
is monadic.

Remark 7.21. The resulting equivalence
$$(\mathrm{Mod}_S)^{\mathrm{op}} = \mathrm{Alg}_T(\mathrm{Mod}_R^{\mathrm{op}}),$$
is much used in algebraic geometry under the name of *faithfully flat descent*, see for example [**StacksProject**] or [**Deligne:Categories**]. One also refers to it by stating it as an equivalence
$$\mathrm{Mod}_S = \mathrm{CoAlg}_T(\mathrm{Mod}_R),$$
where the definition of a *comonad*, and *coalgebras* over a comonad are obtained by reversing all arrows in the definition of a monad and its algebras.

Remark 7.22. The condition that S is faithfully flat not quite necessary for the adjunction to be comonadic. See [**JanelidzeTholen:FacetsIII**] for a necessary and sufficient condition.

We prepare for the proof of the crude monadicity theorem. For a ring R and an R-module M, the natural map
$$\bigoplus_{m \in M} R \to M, \sum_{m \in M} r_m e_m \mapsto \sum_{m \in M} r_m \cdot m$$
is surjective. A massive generalization of this fact, and also a description of the relations in $R[M]$ is provided by the following lemma.

Lemma 7.23. Let $T : C \to C$ be a monad and $(A, \alpha : TA \to A) \in \mathrm{Alg}_T$. Then the following is a reflexive coequalizer diagram in Alg_T:

$$\begin{tikzcd} T(T(A)) \arrow[r, shift left, "T(\alpha)"] \arrow[r, shift right, "\mu_A"'] & T(A) \arrow[l, bend left, "T(\eta_A)"] \arrow[r, "\alpha"] & (A, \alpha). \end{tikzcd}$$

Here, $T(X)$ denotes the free T-algebra (previously denoted $\mathrm{Free}(X)$) on an object $X \in C$. If we forget the T-algebra structures on all objects, the diagram is a reflexive coequalizer diagram in C.

Proof. First off, $T\eta_A$ is indeed a splitting by the unitality of the T-algebra (A, α) and the unitality condition for the monad T. For a cocone $B \in \mathrm{Alg}_T$ as depicted, we want to exhibit a unique dotted T-algebra map g in the following diagram:

$$\begin{tikzcd} T(T(A)) \arrow[r, shift left, "T(\alpha)"] \arrow[r, shift right, "\mu_A"'] & T(A) \arrow[r, "\alpha"] \arrow[d, dotted, "Tg"'] \arrow[dr, "f"] & A \arrow[d, dotted, "g"] \\ & T(B) \arrow[r, "\beta"] & B. \end{tikzcd}$$

We momentarily regard the right triangle as a diagram in C (as opposed to Alg_T). The map $g := f \circ \eta_A$ makes the right triangle commute. Moreover, this is the unique map with this property since $\alpha \circ \eta_A = \mathrm{id}_A$. This directly shows the final coequalizer claim in C. (Alternatively, this also follows from the argument in Example 6.27 below.) For (A, α) to be a coequalizer in Alg_T, we need to still argue as follows: since the forgetful functor $\mathrm{Alg}_T \to C$ is faithful (but not full!), it remains to see that g is indeed a T-algebra map, i.e., the commutativity of the lower right square. We prove this using that f is a T-algebra map (*), the monad axioms of T (**) and the naturality of η (***):

$$\beta \circ Tg := \beta \circ Tf \circ T\eta_A \overset{(*)}{=} f \circ \mu_A \circ T(\eta_A) \overset{(**)}{=} f \overset{(**)}{=} f \circ T(\alpha) \circ \eta_{TA} \overset{(***)}{=} (f \circ \eta_A) \circ \alpha =: g \circ \alpha.$$

Proof. (of Theorem 6.16) We have a factorization of R as

$$D \overset{\tilde{R}}{\to} \mathrm{Alg}_T \overset{U}{\to} C$$

and our goal is to show that $\tilde{R}$ is an equivalence. Our proof is in three steps:
(1) we construct a left adjoint $\tilde{L}$ to $\tilde{R}$,

(2) we show the unit map $\mathrm{id}_{\mathrm{Alg}_T} \to \tilde{R} \circ \tilde{L}$ is a (functorial) isomorphism,

(3) we show that the counit map $\tilde{L} \circ \tilde{R} \to \mathrm{id}_D$ is a (functorial) isomorphism.

(1): if this left adjoint exists, it must satisfy

$$L = \tilde{L} \circ \mathrm{Free} = \tilde{L} \circ T,$$

by Lemma 5.11. I.e., $\tilde{L}(T(X))$ needs to be (isomorphic to) $L(X)$. In addition, $\tilde{L}$ would need to preserve colimits such as the coequalizer in Lemma 6.23. We therefore *define*, for any algebra (A, α), $\tilde{L}(A)$ to fit into the following *reflexive coequalizer* diagram (in D):

$$\begin{tikzcd} L(T(A)) \arrow[r, shift left, "L(\alpha)"] \arrow[r, shift right, "\epsilon_{LA}"'] & L(A) \arrow[l, bend left, "L\eta_A"] \arrow[r, "\theta"] & \tilde{L}(A, \alpha). \end{tikzcd} \tag{7.24}$$

(Here ϵ_{LA} is the evaluation of the counit $\epsilon : LR \to \mathrm{id}_D$ at $LA \in D$). The coequalizer, $\tilde{L}(A, \alpha)$, exists by assumption on D. In order to check $\tilde{L}$ is a left adjoint to $\tilde{R}$, it suffices to exhibit a bijection

$$\mathrm{Hom}_D(\tilde{L}(A, \alpha), B) = \mathrm{Hom}_{\mathrm{Alg}_T}((A, \alpha), \tilde{R}(B))$$

functorially in $B \in \mathrm{Alg}_T$ (Lemma 5.17). Indeed the left hand Hom is $\{f : L(A) \to B, f \circ L(\alpha) \overset{(*)}{=} f \circ \epsilon_{L(A)}\}$. The condition (*) is equivalent to the commutativity of the left hand diagram, which in its turn is equivalent to the commutativity of the diagram at the right, which is obtained by passing to transposes (f^t denotes the transpose of t):

$$\begin{tikzcd} LTA = LRLA \arrow[r, "L\alpha"] \arrow[d, "\epsilon_{LA}"] & LA \arrow[d, "f"] \\ LA \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} RLA \arrow[r, "\alpha"] \arrow[d, equal] & A \arrow[d, "f^t"] \\ RLA \arrow[r, "Rf"] & RB. \end{tikzcd}$$

By definition of the counit map and the transpose, we have a commutative diagram

$$\begin{tikzcd} LA \arrow[r, "L(f^t)"] \arrow[dr, "f"'] & LRB \arrow[d, "\epsilon_B"] \\ & B. \end{tikzcd}$$

Thus, condition (*) is equivalent to $f^t \circ \alpha = R(\epsilon_B)R(L(f^t))$. Thus the above Hom-set is the same as a map of algebras

$$f^t : (A, \alpha) \to \tilde{R}(B) = (RB, R\epsilon_B).$$

(2): we apply R to (6.24). By assumption on R, this yields a coequalizer diagram

$$RL(T(A)) \underset{R\epsilon_{LA}}{\overset{RL(\alpha)}{\rightrightarrows}} RL(A) \xrightarrow{R(\theta)} R\tilde{L}A. \tag{7.25}$$

Note that $RLT = T^2$. We also have the free resolution coequalizer (Lemma 6.23), which we at this point regard as a coequalizer in C (as opposed to Alg_T)

$$T(T(A)) \underset{\mu_A = R\epsilon_{LA}}{\overset{RL(\alpha)}{\rightrightarrows}} T(A) \xrightarrow{\alpha} A.$$

Thus there is a (unique) isomorphism fitting into the following commutative diagram:

$$\begin{tikzcd} RLA \arrow[r, "R\theta"] \arrow[dr, "\alpha"'] & R\tilde{L}(A, \alpha) \arrow[d, "\cong"] \\ & A. \end{tikzcd}$$

Henceforth we will identify A with $R\tilde{L}(A, \alpha)$ using this isomorphism.

In order to complete our claim, we need to show that the T-algebra structures on $R\tilde{L}(A, \alpha)$ and on A agree, i.e., $R\epsilon_{\tilde{L}(A,\alpha)} = \alpha$. Since $\alpha = R\theta$, our claim follows from the following computation, where we use the unitality condition of the T-algebra (A, α) (a), the definition of θ as a coequalizer in (6.24) (b), and the naturality of ϵ (c) and, in (d), the unitality of the T-algebra $R\tilde{L}(A, \alpha)$:

$$\theta \overset{(a)}{=} \theta \circ L\alpha \circ L\eta_A \overset{(b)}{=} \theta \circ \epsilon_{LA} \circ L\eta_A \overset{(c)}{=} \epsilon_{\tilde{L}(A,\alpha)} \circ LR(\theta) \circ L\eta_A \overset{(d)}{=} \epsilon_{\tilde{L}(A,\alpha)}.$$

(3): Applying the definition of $\tilde{L}$ in (6.24) to $A = \tilde{R}(B)$, we get the following reflexive coequalizer diagram:

$$\begin{tikzcd} LRLRB \arrow[r, shift left, "LR\epsilon_B"] \arrow[r, shift right, "\epsilon_{LRB}"'] & LR(B) \arrow[l, bend left, "L\eta_{RB}"] \arrow[r, "\theta"] \arrow[dr, "\epsilon_B"'] & \tilde{L}\tilde{R}(B) \arrow[d, dotted, "\tau"] \\ & & B. \end{tikzcd}$$

The dotted map τ exists by the universal property of the colimit, noting that $\epsilon_B \circ LR\epsilon_B$ and $\epsilon_B \circ \epsilon_{LRB}$ are both transposes of $R\epsilon_B$, and therefore agree. The map $L\eta_{RB}$ is a splitting of the two maps by the triangle identities. By assumption on R, applying R, we still have a (reflexive) coequalizer diagram:

$$\begin{tikzcd} RLRLRB \arrow[r, shift left, "RLR\epsilon_B"] \arrow[r, shift right, "R\epsilon_{LRB}"'] & RLR(B) \arrow[r, "R\theta"] \arrow[dr, "R\epsilon_B"'] & R\tilde{L}\tilde{R}(B) \arrow[d, dotted, "R\tau"] \\ & & RB. \end{tikzcd}$$

The two parallel maps and $R\epsilon_B$ also form a coequalizer by Lemma 6.23. Thus, $R\tau$ is an isomorphism. By assumption on R, τ is therefore an isomorphism as well. ☐

The above "crude" monadicity is not quite necessary, but we have done almost all the work to get a necessary (and sufficient) condition for an adjunction to be monadic.

Definition and Lemma 7.26. A *split coequalizer diagram* in a category C is a diagram of shape

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left=50, "t"] \arrow[r, "e"] & C, \arrow[l, bend left=50, "s"] \end{tikzcd}$$

with $ef = eg$ and the "wrong way maps" satisfy $es = \mathrm{id}_C$, $se = gt$ and $ft = \mathrm{id}_B$.

For such a diagram, C (together with the map e) indeed is a coequalizer of (f, g). *Any* functor preserves split coequalizer diagrams. (A colimit that is preserved by any functor is called an *absolute colimit*.)

Proof. Any map $k : B \to K$ such that $k \circ f = k \circ g$ factors over e:

$$k = kft = kgt = kse.$$

Example 7.27. Revisiting Lemma 6.23, we note that for a monad $T : C \to C$, and a T-algebra, the diagram is a split coequalizer diagram in C

$$\begin{tikzcd} T(T(A)) \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\alpha"'] & T(A) \arrow[l, bend left=50, "\eta_{TA}"] \arrow[r, "\alpha"] & A. \arrow[l, bend left=50, "\eta_A"] \end{tikzcd}$$

(Parallely the diagram is also a reflexive coequalizer diagram in Alg_T, but not a split coequalizer diagram in Alg_T: the map η_{TA} is not a map of T-algebras.)

Theorem 7.28. (*Precise monadicity theorem*) An adjunction $L \dashv R$ is monadic if and only if

- D admits and R preserves coequalizers of R-split pairs. By this we mean that for any pair of maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in D such that $RX \underset{Rg}{\overset{Rf}{\rightrightarrows}} RY$ can be extended to a split coequalizer diagram (in C), then the original diagram admits a colimit, and R preserves that colimit.
- R is conservative.

This can be proven along the same lines as the crude monadicity theorem above.

Corollary 7.29. The adjunction

$$\beta : \mathrm{Set} \rightleftarrows \mathrm{CptHaus} : U$$

is monadic.

Proof. For the proof, we use that a topological space is the same as a set X and a map (the *closure*) $\overline{} : PX \to PX$ such that

$$\overline{\varnothing} = \varnothing, \overline{S \cup T} = \overline{S} \cup \overline{T},\ S \subset \overline{S},\ \overline{\overline{S}} = \overline{S}.$$

A continous map is a map such that $f(\overline{S}) \subset \overline{f(S)}$.

Let $X, Y \in \mathrm{CptHaus}$, f and g be continuous maps such that for the underlying sets, there is a split coequalizer diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend left, "t"] \arrow[r, "e"] & W, \arrow[l, bend left, "s"] \end{tikzcd}$$

We can apply the covariant power set functor (Example 6.6). Since split coequalizer diagrams are preserved by any functor, the rows are still coequalizers. The maps f and g are continuous and, being maps between compact Hausdorff spaces, necessarily closed maps. Thus, the left square commutes, so that there is a unique map (of sets), $\overline{?}$ as depicted at the right:

$$\begin{tikzcd} PX \arrow[r, shift left, "Pf"] \arrow[r, shift right, "Pg"'] \arrow[d, "\overline{?}"] & PY \arrow[r, "Pe"] \arrow[d, "\overline{?}"] & PW \arrow[d, dotted, "\overline{?}"] \\ PX \arrow[r, shift left, "Pf"] \arrow[r, shift right, "Pg"'] & PY \arrow[r, "Pe"] & PW \end{tikzcd} \tag{7.30}$$

Note that for $T \subset W$, $\overline{T} = e(\overline{e^{-1}T})$. Using this, it is trivial to check that the four conditions on the closure are satisfied. Thus, W is a topological space, and the map e is continuous. What is more, a subset $T \subset W$ is closed iff $e^{-1}(T) \subset Y$ is closed (since in the latter case $T = e(e^{-1}(T))$ is closed by the commutativity of the right half in (6.30)). Thus, the topology on W is the *quotient topology*.

Since Y is compact, and e is surjective, W is also compact. Let $w_1 \neq w_2 \in W$ be distinct points. We have $s(w_1) \neq s(w_2) \in Y$ (since $es = \mathrm{id}_W$), so there are open disjoint neighborhoods $U_i \ni s(w_i)$ in Y. Their complements $Z_i := Y \backslash U_i$ are closed and cover Y. The map e is closed since the right part in (6.30) commutes. Thus $e(Z_i)$ are closed, and cover $W = e(Y)$. Their complements $W \backslash e(Z_i)$ are thus disjoint open neighborhoods of w_i, so that W is Hausdorff.

We finally check that the compact Hausdorff space W is indeed a coequalizer. Let $h : Y \to Z$ be a continuous map with $hf = hg$. Since W is a coequalizer in Set, h factors over e: $h = h'e$. Since W carries the quotient topology, h' is continuous as soon as h is. □

Remark 7.31. The above statement leads to the question of describing the monad associated to the adjunction $\beta \dashv U$ more concretely. One can show (without much difficulty) that the monad is the so-called *ultrafilter monad*[1]

$$T : \mathrm{Set} \to \mathrm{Set}, X \mapsto \{\text{ultrafilters on } X\}.$$

Note that the following diagram commutes:

$$\begin{tikzcd} \mathrm{FinSet} \arrow[r] \arrow[d, "\iota"] & \mathrm{CptHaus} = \mathrm{Alg}_T \arrow[d, "U"] \\ \mathrm{Set} \arrow[r, equal] & \mathrm{Set}. \end{tikzcd}$$

Note that ι is *not* part of an adjunction (Exercise 5.17). Nonetheless, one can associate a monad to this inclusion, called the *codensity monad*, which happens to be the ultrafilter monad. This leads to a characterization of compact Hausdorff spaces in completely categorical terms: one can show (again without much difficulty) that it is the initial monadic adjunction over Set fitting into a diagram like so:

$$\begin{tikzcd} \mathrm{FinSet} \arrow[r] \arrow[d, "\iota"] & C \arrow[d, "R"] \\ \mathrm{Set} \arrow[r, equal] & \mathrm{Set}. \end{tikzcd}$$

See [**Leinster:Codensity**] for an exposition of these ideas.

[1] An ultrafilter on a set X is a collection of subsets $V \subset X$ that is upwards closed and stable under intersection, and for any subset $W \subset X$ either W or $X \backslash W$ (but not both) are in the collection.

This understanding of CptHaus can also be used to prove Riesz' representation theorem [**Hartig:Riesz**], stating that for any compact Hausdorff space X, there is a (functorial!) isomorphism (of Banach spaces)

$$M(X) \to C(X)^*, \mu \mapsto (f \mapsto \int_X f d\mu)$$

between the space of *measures* on X and the dual space of the space of continuous, real-valued functions on X. The basic idea in this categorical proof is to observe that it is enough to prove this for projective T-algebras (which are classically known as *extremally disconnected spaces*), i.e., retracts of algebras of the form $\beta(X)$, for $X \in \mathrm{Set}$. For these, one checks the result by an immediate inspection.

Finally, extremally disconnected spaces play an outsize rôle in Clausen–Scholze's foundational work on condensed mathematics [**Scholze:Lectures**].

7.5 Exercises

Exercise 7.1. Let $T : C \to C$ be a monad on a category. Show that the Kleisli category for T is equivalent to the category D with $\mathrm{Obj}(D) = \mathrm{Obj}(C)$ and with morphisms given by

$$\mathrm{Hom}_D(X, Y) := \mathrm{Hom}_C(X, T(Y))$$

and an appropriate composition.

Exercise 7.2. Let Set_* be the category of *pointed sets*, i.e., pairs (X, x) with a so-called base-point $x \in X$ and maps preserving the base point. (More succinctly, the category can be defined as a comma category: $\mathrm{Set}_* := \{*\}/\mathrm{Set}$.)

(1) Show that the forgetful functor $U : \mathrm{Set}_* \to \mathrm{Set}$ has a left adjoint L.

(2) Show that the adjunction $L \dashv U$ is monadic.

(3) Let $T := UL$ be the monad associated to the adjunction. Show that the Kleisli category of T is equivalent to the category $\mathrm{Set}^{\mathrm{part}}$ of *sets with partial maps*, i.e.,

$$\mathrm{Hom}_{\mathrm{Set}^{\mathrm{part}}}(X, Y) = \{X' \subset X, f \in \mathrm{Hom}_{\mathrm{Set}}(X', Y)\}.$$

Exercise 7.3. Imitate the proof of Theorem 6.16 to show Theorem 6.28.

Exercise 7.4. Apply Theorem 6.28 to show that the free-forgetful adjunction

$$\mathrm{Free} : \mathrm{Set} \rightleftarrows \mathrm{Grp} : U$$

is monadic.

Hint: assume that $G \overset{f}{\underset{g}{\rightrightarrows}} H$ is a parallel pair of arrows that is split in sets (as in Definition and Lemma 6.26). Show that $G \times G \overset{f \times f}{\underset{g \times g}{\rightrightarrows}} H \times H$ is also part of a U-split coequalizer diagram. Then mimic the situation in (6.30), with the group operation $G \times G \to G$ instead of the closure operator.

Chapter 8

Kan extensions

Given a functor $F : C \to C'$ and some category E, there is an obvious functor

$$F^* : \mathrm{Fun}(C', E) \to \mathrm{Fun}(C, E), H \mapsto H \circ F.$$

Kan extensions, more precisely left Kan extensions and right Kan extensions are a way to go backwards: given a functor $G : C \to E$, they produce (under certain conditions) two functors

$$\mathrm{Lan}_F(G), \mathrm{Ran}_F(G) : C' \to E.$$

We will depict this situation by

$$\begin{tikzcd} C \arrow[r, "G"] \arrow[d, "F"'] & E \\ C'. \arrow[ur, dotted, "\mathrm{Lan}_F G, \mathrm{Ran}_F G"'] & \end{tikzcd}$$

The precise condition how these functors $\mathrm{Lan}_F G$ and $\mathrm{Ran}_F G$ are related to G is given by an adjunction:

Definition 8.1. A *left Kan extension* of G along F, denoted by $\mathrm{Lan}_F(G)$ (dually, a *right Kan extension*, $\mathrm{Ran}_F(G)$) is a functor $C' \to E$, i.e., an object in $\mathrm{Fun}(C', E)$, such that there is a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}(C',E)}(\mathrm{Lan}_F(G), -) \cong \mathrm{Hom}_{\mathrm{Fun}(C,E)}(G, F^* -),$$

(respectively,

$$\mathrm{Hom}_{\mathrm{Fun}(C',E)}(-, \mathrm{Ran}_F(G)) \cong \mathrm{Hom}_{\mathrm{Fun}(C,E)}(F^* -, G).)$$

Example 8.2. Let $f : \mathbf{Q} \to \mathbf{R}$ be a monotone function (i.e., $f(x) \leqslant f(y)$ for $x \leqslant y$). We regard f as a functor, by regarding $\mathbf{Q}$ and $\mathbf{R}$ as posets (and therefore categories) via the usual relation $\leqslant$. Then the supremum $\sup_{s \leqslant x, s \in \mathbf{Q}} f(s)$ exists for each $x \in \mathbf{R}$ (using that $\mathbf{R}$ is complete, and since the $f(s)$ are bounded by $f(t)$ for some $t \in \mathbf{Q}$, $t \geqslant x$), then the function

$$\overline{f} : \mathbf{R} \to \mathbf{R}, x \mapsto \sup_{s \leqslant x} f(s)$$

is a left Kan extension of f along the inclusion $\mathbf{Q} \subset \mathbf{R}$. (Dually, $x \mapsto \inf_{s \geqslant x} f(s)$ is a right Kan extension.) Indeed, given a monotone function $h : \mathbf{R} \to \mathbf{R}$, $\mathrm{Hom}_{\mathrm{Fun}(\mathbf{Q},\mathbf{R})}(f, h|_{\mathbf{Q}})$ is a singleton iff $f(s) \leqslant h(s)$ for each $s \in \mathbf{Q}$, and is empty otherwise. The former is equivalent to

$$\overline{f}(x) = \sup_{s \leqslant x} f(s) \leqslant h(x)$$

for each $x \in \mathbf{R}$. Then, $\Rightarrow$ follows by taking sup, and $\Leftarrow$ follows by taking $x = s$.

A middle-school application of this procedure is the construction of the exponential function, for $a \geqslant 1$

$$f(x) := a^x : \mathbf{R} \to \mathbf{R}, x \mapsto a^x,$$

which can be defined as $a^x := \sup_{s \leqslant x} a^s$, provided that powers with rational exponents are already defined. (For this particular function f, right and left Kan extension happen to agree, which is unusual for general Kan extensions.)

This example indicates that (right or left) Kan extensions need not exist. We will prove (Proposition 8.15) that left Kan extensions (along a functor between small categories) do exist as soon as the target category E is cocomplete. We will use Kan extensions to formulate (and prove) the universal property of the category $\widehat{C} = \mathrm{PSh}(C)$ of presheaves: it is the free cocompletion (Theorem 8.20). This cocompletion is also useful in order to construct free completions for more restrictive colimits, notably the Ind-completion, which freely adds filtered colimits.

Another prominent application of Kan extensions are *derived functors*. For example, the functor

$$\mathbf{F}_p \otimes_{\mathbf{Z}} - : \mathrm{Ab} \to \mathrm{Mod}_{\mathbf{F}_p}$$

which is right exact, but not left exact admits a so-called left derived functor, which measures the deviation of this functor to be left exact (and therefore exact). This left derived tensor product is the right Kan extension in the following diagram:

$$\begin{tikzcd} \mathrm{Ch}(\mathrm{Ab}) \arrow[r, "\mathbf{F}_p\otimes -"] \arrow[d, "c"] & \mathrm{Ch}(\mathrm{Mod}_{\mathbf{F}_p}) \arrow[d, "c"] \\ \mathrm{D}(\mathrm{Ab}) \arrow[r, "\mathbf{F}_p\otimes^L_{\mathbf{Z}} -"] & \mathrm{D}(\mathrm{Mod}_{\mathbf{F}_p}). \end{tikzcd}$$

Here Ch denotes the category of chain complexes, and D the so-called *derived category*, which is the category of chain complexes with quasi-isomorphisms inverted. See [Mac98, §X.4] for a glimpse or [**KahnMaltsiniotis:Stru** for a full-fledged exposition in greater generality.

Remark 8.3. The name Kan *extension* may suggest the idea

$$\mathrm{Lan}_F G \circ F = G.$$

This statement holds true if F is fully faithful (Corollary 8.19), but not necessarily otherwise: Kan extensions along $C \to \{*\}$ are (co)limits, so that, in particular, the triangle does not commute, unless G is a constant functor.

$$\begin{tikzcd} C \arrow[r, "G"] \arrow[d, "F"] & E \\ \{*\} \arrow[ur, "\mathrm{Lan}_F G = \mathrm{colim}\, G, \mathrm{Ran}_F G = \lim G"'] & \end{tikzcd}$$

Indeed, functors $\{*\} \to E$ are just objects in e and the functor $F^* : E \to \mathrm{Fun}(C, E)$ is simply the diagonal functor previously denoted Δ, so the claim follows from Lemma 5.10.

8.1 (Co)ends

Our construction of right Kan extensions below will be based on ends (resp. coends for left Kan extensions). These are special limits (resp. colimits) over the twisted arrow category, all of which we introduce in this section.

Definition 8.4. Let C be a small category. Define the *subdivision category* $C^\S$ to have as objects symbols $c^\S$ and $f^\S$, for each object $c \in \mathrm{Obj}(c)$ and each morphism $f : c \to c'$ in C. (By definition, $c^\S \neq (\mathrm{id}_c)^\S$.) By definition, any morphism in this category is a composition of identity morphisms of these objects, together with morphisms of the form

$$c^\S \to f^\S, c'^\S \to f^\S \tag{8.5}$$

for each $f : c \to c'$.

There is a natural functor

$$\iota : C^\S \to C^{\mathrm{op}} \times C,$$

given on objects by $c^\S \mapsto (c, c)$, $(f : c \to c') \mapsto (c, c')$ and on morphisms in the expected way: $c^\S \to f^\S$ is mapped to (id_c, f), and $c'^\S \to f^\S$ is mapped to $(f, \mathrm{id}_{c'})$.

Definition 8.6. Let $F : C^{\mathrm{op}} \times C \to D$ be a functor. An *end* (resp. *coend*) of F is a limit (resp. colimit) of the composition

$$C^{\S} \xrightarrow{\iota} C^{\mathrm{op}} \times C \xrightarrow{F} D.$$

An end is denoted by $\int_c F(c,c)$, and a coend by $\int^c F(c,c)$.

Concretely, this means that the end $\int_c F(c,c)$ is an object $d \in D$ together with maps

$$r_c : d \to F(c,c), r_f : d \to F(c,c')$$

(for each $f : c \to c'$ in C) such that the diagram

$$\begin{tikzcd}
d' \arrow[r] \arrow[drr] \arrow[ddr] \arrow[d, dotted] & F(c,c) \arrow[dr, "F(c,f)"] & \\
d \arrow[ur, "r_c"] \arrow[rr, "r_f"] \arrow[dr, "r_{c'}"] & & F(c,c') \\
& F(c',c'). \arrow[ur, "F(f,c)"] &
\end{tikzcd}$$

commutes. Moreover, for any other object d' with the same properties, there is a unique map $d' \to d$ making the whole diagram commutative.

Equivalently, an end can be computed as an equalizer

$$\int_c F(c,c) = \mathrm{eq}\left(\prod_{c \in C} F(c,c) \rightrightarrows \prod_{f : c \to c'} F(c,c')\right), \tag{8.7}$$

where for each morphism $f : c \to c'$, the two maps are given by

- $\prod_c F(c,c) \to F(c,c) \overset{F(c,f)}{\to} F(c,c')$,
- $\prod_c F(c,c) \to F(c',c') \overset{F(f,c')}{\to} F(c,c')$.

Remark 8.8. The above definition of $C^{\S}$ is somewhat ad hoc. A slightly more conceptual approach is to define the *twisted arrow category* $\mathrm{Tw}(C)$ to be the category whose objects are the morphisms in C. For two morphisms $f : A \to A'$, $g : B \to B'$ in C (i.e., objects in $\mathrm{Tw}(C)$, we define $\mathrm{Hom}_{\mathrm{Tw}(C)}(f,g)$ to be the set of commutative squares

$$\begin{tikzcd}
A \arrow[d, "f"] & B \arrow[l, "h"'] \arrow[d, "g"] \\
A' \arrow[r, "k"] & B'.
\end{tikzcd} \tag{8.9}$$

(Note the arrow h goes in the “wrong” direction.) The functor ι above factors as a composite

$$C^{\S} \to \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C,$$

the former functor being given by $c^{\S} \mapsto \mathrm{id}_c$, and morphisms in (8.5) are mapped to

$$\begin{tikzcd}
c' \arrow[d, "\mathrm{id}'_c"'] & c \arrow[l, "f"'] \arrow[d, "f"] \\
c' \arrow[r, "\mathrm{id}'_c"] & c'
\end{tikzcd}, \quad
\begin{tikzcd}
c \arrow[d, "\mathrm{id}_c"'] & c \arrow[l, "\mathrm{id}_c"'] \arrow[d, "f"] \\
c \arrow[r, "f"] & c'
\end{tikzcd}. \tag{8.10}$$

The second functor is given on objects by $(f : c \to c') \to (c, c')$.

One checks that the (co)limit of the composite

$$C^{\S} \to \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C \xrightarrow{F} D$$

and the (co)limit of

$$\mathrm{Tw}(C) \to C^{\mathrm{op}} \times C \xrightarrow{F} D$$

are isomorphic, see e.g. [**Richter:Categories**]. Thus, (co)ends can be defined, somewhat less ad hoc, in the latter way.

Example 8.11. Let $A, B \in \mathrm{Fun}(C, D)$ be two functors, where C is small. Consider the functor

$$F : C^{\mathrm{op}} \times C \xrightarrow{A \times B} D^{\mathrm{op}} \times D \xrightarrow{\mathrm{Hom}_D(-,-)} \mathrm{Set}, (c, c') \mapsto \mathrm{Hom}_D(A(c), B(c')).$$

The end of this functor recovers the set of natural transformations between A and B:

$$\int_c F(c,c) = \int_c \mathrm{Hom}_D(A(c), B(c)) = \mathrm{Hom}_{\mathrm{Fun}(C,D)}(A, B). \tag{8.12}$$

Indeed, a morphism $\alpha : A \to B$ is a collection of elements in $\alpha_c \in \mathrm{Hom}_D(A(c), B(c)) = F(c,c)$, subject to the condition that the diagrams

$$\begin{tikzcd} A(c) \arrow[r, "A(f)"] \arrow[d, "\alpha_c"] & A(c') \arrow[d, "\alpha_{c'}"] \\ B(c) \arrow[r, "B(f)"] & B(c') \end{tikzcd}$$

commutes. This is precisely the content of the equalizer in (8.7).

In the sequel, we will often have to talk about cocontinuous functors, i.e., colimit-preserving functors, so we introduce a notation:

Definition 8.13. Let I be a small category and C a cocomplete category. Then $\mathrm{Fun}^{\mathrm{c}}(I, C)$ is the full subcategory of the functor category $\mathrm{Fun}(I, C)$ consisting of functors preserving all (existing) colimits.

Let E be a category with all (small) coproducts. Recall that by Exercise 5.18 the representable functor $\mathrm{Hom}_E(e, -) : E \to \mathrm{Set}$ admits a left adjoint, denoted $- \otimes e$, given on objects by

$$s \otimes e := \coprod_{* \in s} e,$$

a coproduct indexed by the elements of the set $s \in \mathrm{Set}$. Being a left adjoint, this functor $- \otimes e$ preserves colimits, and is thus an object in $\mathrm{Fun}^{\mathrm{c}}(\mathrm{Set}, E)$. By Exercise 5.4, for any cocomplete category E, there is an equivalence

$$\mathrm{Fun}^{\mathrm{c}}(\mathrm{Set}, E) \subset \mathrm{Fun}(\mathrm{Set}, E) \xrightarrow{F \mapsto F(\{*\})} E,$$

with an inverse functor given by $e \mapsto (- \otimes e)$.

The following related statement can be viewed as a category-theoretic version of the formula

$$\mu(A) = \int_X \chi_A(x) d\mu(x),$$

for a measure space (X, μ) and a measurable subset $A \subset X$. It expresses a decomposition of a functor F into two independent pieces: the values $F(i)$ and the Hom-sets $\mathrm{Hom}(-, i)$.

Lemma 8.14. Let I be a small category, and D be cocomplete. For a functor $F : I^{\mathrm{op}} \to D$, there is a functorial isomorphism

$$F \cong \int^i \mathrm{Hom}(-, i) \otimes F(i).$$

Proof. We need to show that $F(j)$ is (functorially) isomorphic to the colimit of the diagram

$$I^{\S} \xrightarrow{\iota} I^{\mathrm{op}} \times I \xrightarrow{F\times\mathrm{id}} D \times I \xrightarrow{\mathrm{id}\times\mathrm{Hom}_I(j,-)} D \times \mathrm{Set} \xrightarrow{\otimes} D.$$

A cocone over that diagram is an object $d \in D$ together with a collection of maps τ_i (for each $i \in I$), such that for each map $i \to i'$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(i',j) \otimes F(i) \arrow[r] \arrow[d, "F(f)"] & \mathrm{Hom}(i,j) \otimes F(i) \arrow[d, "\tau_i"] \\
\mathrm{Hom}(i',j) \otimes F(i') \arrow[r, "\tau_{i'}"] & d.
\end{tikzcd}$$

By definition of $\otimes$ as a left adjoint, the map τ_i is equivalent to a family of maps $\tau_{i,g} : F(i) \to d$ for each $g \in \mathrm{Hom}(i,j)$, and then the condition amounts to requiring that these assemble to a morphism of functors (in $\mathrm{Fun}(I^{\mathrm{op}}, \mathrm{Set})$)

$$\mathrm{Hom}_I(-,j) \to \mathrm{Hom}_D(F(-),d), (g : i \to j) \mapsto (\tau_{i,g} : F(i) \to d).$$

By the Yoneda lemma (Lemma 3.5), this is the same as the evaluation of the right hand functor at j, i.e., a morphism $F(j) \to d$, namely τ_{j,id_j}. Thus, the initial cocone is $F(j)$, showing our claim. □

8.2 Kan extensions via (co)ends

Proposition 8.15. Let E be a *cocomplete* category, and $F : C \to C'$ be a functor between *small* categories. Then the restriction functor

$$F^* : \mathrm{Fun}(C', E) \to \mathrm{Fun}(C, E)$$

admits a left adjoint. That is, all functors $G : C \to E$ have a left Kan extension along F. It can be computed as the coend of the functor

$$\mathrm{Lan}_F(G) = \int^c \mathrm{Hom}_{C'}(F(c), -) \otimes G(c), \tag{8.16}$$

i.e.,

$$(\mathrm{Lan}_F(G))(c') = \int^c \mathrm{Hom}_{C'}(F(c), c') \otimes G(c).$$

Example 8.17. If $F = \mathrm{id}_C$, then F^* is just the identity so that its left adjoint, $\mathrm{Lan}_F(-)$ is also the identity, i.e., $\mathrm{Lan}_F(G) = G$. Thus, the statement here can be regarded as an extension of Lemma 8.14.

At another extreme, for $F : C \to \{*\}$, we already know $\mathrm{Lan}_F G = \operatorname{colim} G$. For this reason, the coend appearing in (8.16) is referred to as a *weighted colimit*, the idea being that the terms $G(c)$ appearing in the colimit are taken into account with "weight" $\mathrm{Hom}(F(c), c')$.

Proof. There are bijections (functorial in $G \in \mathrm{Fun}(C, E)$ and $H \in \mathrm{Fun}(C', E)$):

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Fun}(C',E)}(\mathrm{Lan}_F G, H) &= \int_{c'} \mathrm{Hom}_E(\mathrm{Lan}_F G(c'), H(c')) \\
&= \int_{c'} \mathrm{Hom}_E\left(\int^c \mathrm{Hom}_{C'}(F(c), c') \otimes G(c), H(c')\right) \\
&= \int_{c'} \int_c \mathrm{Hom}_E\left(\mathrm{Hom}_{C'}(F(c), c') \otimes G(c), H(c')\right) \\
&= \int_{c'} \int_c \mathrm{Hom}_{\mathrm{Set}}\left(\mathrm{Hom}_{C'}(F(c), c'), \mathrm{Hom}_E(G(c), H(c'))\right) \\
&\overset{*}{=} \int_c \int_{c'} \mathrm{Hom}_{\mathrm{Set}}\left(\mathrm{Hom}_{C'}(F(c), c'), \mathrm{Hom}_E(G(c), H(c'))\right) \\
&\overset{**}{=} \int_c \mathrm{Hom}_E(G(c), H(F(c)) \\
&= \mathrm{Hom}_{\mathrm{Fun}(C,E)}(G, H \circ F) \\
&= \mathrm{Hom}_{\mathrm{Fun}(C,E)}(G, F^* H).
\end{aligned}$$

We have used (8.12), the above definition, the cocontinuity of $\mathrm{Hom}_E(-, H(c'))$, the definition of the $\otimes$-functor as a left adjoint. At the end, we have again used (8.12), and the definition of F^*.

The isomorphism * is an instance of the Fubini theorem for ends, which is an extension of Lemma 4.44: let us be given a functor

$$F : C^{\mathrm{op}} \times C \times C'^{\mathrm{op}} \times C' \to V$$

with V complete (for example, $V = \mathrm{Set}$; it suffices to assume that all the ends appearing below exist). Then, taking the end with respect to C gives a functor

$$\int_c F : C'^{\mathrm{op}} \times C' \to V, (c'_1, c'_2) \mapsto \int_c F(c, c, c'_1, c'_2).$$

The end of this functor is then isomorphic to the end of F, regarded as a functor $(C \times C')^{\mathrm{op}} \times (C \times C') \to V$. In particular, by symmetry, we then have

$$\int_c \int_{c'} F(c, c, c', c') = \int_{c'} \int_c F(c, c, c', c').$$

For a hands-on proof of this, see [Mac98, Proposition §IX.8].

At **, we have used (8.12) again: if we put $X(c') := \mathrm{Hom}_E(G(c), H(c'))$, then an element in the inner coend is a compatible collection, for each $F(c) \to c'$, of elements of $X(c')$. This is the same as an element in $X(F(c))$. □

Remark 8.18. Another proof of the Fubini theorem for coends proceeds by observing that the process of taking coends is a left adjoint, much the same way as colimits are left adjoints (Lemma 5.10): for a cocomplete category E, there is an adjunction

$$\int_c : \mathrm{Fun}(C^{\mathrm{op}} \times C, E) \rightleftarrows E,$$

where the right adjoint is given by $e \mapsto ((c, c') \mapsto \mathrm{Hom}_C(c, c') \otimes e)$, see [**Loregian:Coend**]. Using this insight, the Fubini theorem for coends can be conceptually proven as we did for colimits, cf. p. **??**.

Corollary 8.19. In the situation of Proposition 8.15, suppose also that F is fully faithful. Then we have

$$(\mathrm{Lan}_F G)|_C = G.$$

Proof. For $c' \in C \subset C$, we have by Lemma 8.14:

$$\mathrm{Lan}_F G(c') = \int^c \mathrm{Hom}_{C'}(F(c), c') \otimes G(c) = \int^c \mathrm{Hom}_C(c, c') \otimes G(c) = G(c').$$

8.3 The free cocompletion

The following statement is referred to by saying that the presheaf category $\widehat{C} := \mathrm{PSh}(C)$ is the free cocompletion of a (small) category C. Another way to phrase the result is by saying that any functor (taking values in a cocomplete category D) can be uniquely (up to unique isomorphism) extended to a colimit-preserving functor defined on the presheaf category. This extension is the left Kan extension:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "y"'] & D \\ \widehat{C} \arrow[ur, "\mathrm{Lan}_y F"'] & \end{tikzcd}$$

Theorem 8.20. Let C be a small category and $y : C \to \widehat{C}$ the Yoneda embedding into its category of presheaves. For any cocomplete category D, the composite

$$\mathrm{Fun}^{\mathrm{c}}(\widehat{C}, D) \subset \mathrm{Fun}(\widehat{C}, D) \xrightarrow{y^*} \mathrm{Fun}(C, D)$$

is an equivalence of categories, with an inverse given by left Kan extensions.

Example 8.21. Let $f : X \to Y$ be a continuous map of topological spaces, and consider the associated functor

$$f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X).$$

Composition with this functor defines the so-called *pushforward functor*

$$f_* : \mathrm{PSh}(X) := \mathrm{PSh}(\mathrm{Open}(X)) \to \mathrm{PSh}(\mathrm{Open}(Y)).$$

Concretely, for a presheaf F on X, f_*F is the presheaf given by $(f_*F)(U) = F(f^{-1}(U))$. Consider the following left Kan extension:

$$\begin{tikzcd} \mathrm{Open}(X) \arrow[r, "f^{-1}"] \arrow[d, "y_X"] & \mathrm{Open}(Y) \arrow[d, "y_Y"] \\ \mathrm{PSh}(X) \arrow[r, "f^* := \mathrm{Lan}_{y_X}(y_Y \circ f^{-1})"'] & \mathrm{PSh}(Y). \end{tikzcd}$$

The functor f^* is called the *pullback functor*. According to the theorem, it is the unique colimit preserving functor that is given on representable presheaves by

$$f^*(\mathrm{Hom}(-, U)) = \mathrm{Hom}(-, f^{-1}(U)).$$

One can further show that f^* is a left adjoint of f_* (Exercise 8.3), which makes this functor into a crucial functor in geometry and topology.

If $C = \{*\}$, then Theorem 8.20 recovers Exercise 5.4. We will in fact prove Theorem 8.20 as a consequence of Exercise 5.4 and some considerations on Kan extensions. Recall that for two small categories I and J and a category C, we have a natural equivalence

$$\mathrm{Fun}(I \times J, C) \cong \mathrm{Fun}(I, \mathrm{Fun}(J, C)) \tag{8.22}$$

given by $f \mapsto (i \mapsto (j \mapsto f(i, j)))$ and conversely $f \mapsto ((i, j) \mapsto f(i)(j))$. In other words, the process of taking functor categories resembles being a right adjoint to taking products of categories. (We say "resembles" here, since a regular adjunction involves functorial bijections of certain Hom-sets, whereas the above involves an equivalence of certain categories. The above is an example of an adjunction in the *2-category* of categories, in which the morphisms are not just sets, but categories in their own right.) The following lemma is a noteworthy statement in that it allows to regard the functor category also as a certain *left* adjoint:

Lemma 8.23. Let I, C, D be categories, with I small and C and D cocomplete. Consider the functor

$$\iota : I^{\mathrm{op}} \times C \to \mathrm{Fun}(I, C), (i, c) \mapsto \mathrm{Hom}_I(i, -) \otimes c.$$

Then there is an equivalence of categories

$$\mathrm{Fun}^{\mathrm{c}}(\mathrm{Fun}(I, C), D) \underset{\mathrm{Lan}_\iota -}{\overset{\iota^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{c}_C}(I^{\mathrm{op}} \times C, D) \cong \mathrm{Fun}^{\mathrm{c}}(C, \mathrm{Fun}(I^{\mathrm{op}}, D)).$$

Here, in the middle the superscript c_C indicates the full subcategory consisting of those functors that preserve colimits in the variable C, i.e., those functors such that for each $i \in I$, the functors $F(i, -) : C \to D$ is cocontinuous.

Proof. By Proposition 8.15, Lan_ι exists and can be computed using coends. If $H : I^{\mathrm{op}} \times C \to D$ preserves colimits in C, then $\mathrm{Lan}_\iota H$ is cocontinuous since colimits commute with coends, which are colimits in their own right (**??**). The functor ι preserves colimits in C, so that ι^* gives a functor as stated. Thus, we get pair of functors ι^* and Lan_ι as in the statement.

For the claimed left hand equivalence, we show that the two compositions are isomorphic to the identity functors:

$$\begin{aligned}(\mathrm{Lan}_\iota H)(\iota(j, c)) &= \int^i H(i, \mathrm{Hom}(j, i) \otimes X) \\ &= \int^i \mathrm{Hom}(j, i) \otimes H(i, X) \\ &= H(j, X).\end{aligned}$$

We have used the description of the left Kan extension via coends, and the cocontinuity of H in the second variable, as well as Lemma 8.14.

Conversely, let $\tilde{H}$ be in the left-hand category. Then, being cocontinuous, $\tilde{H}$ preserves coends, and thus, again using Lemma 8.14:

$$\tilde{H}(X) \cong \int^i \tilde{H}(\mathrm{Hom}(i, -) \otimes X(i)) \cong \int^i \tilde{H}\iota(i, X(i)).$$

todo: check

The right hand equivalence arises by restricting the equivalence

$$\mathrm{Fun}(I^{\mathrm{op}} \times C, D) \cong \mathrm{Fun}(C, \mathrm{Fun}(I^{\mathrm{op}}, D))$$

to the indicated subcategories. □

Proof. (of Theorem 8.20) In Lemma 8.23, we take $C = \mathrm{Set}$ and obtain, using Exercise 5.4

$$\mathrm{Fun}^{\mathrm{c}}(\widehat{I}, D) = \mathrm{Fun}^{\mathrm{c}}(\mathrm{Set}, \mathrm{Fun}(I, D)) \xrightarrow[F \mapsto F(\{*\})]{\cong} \mathrm{Fun}(I, D).$$

8.4 The Ind-completion

The category $\mathrm{PSh}(C)$ offers plenty of space for other cocompletions. We will indicate the Ind-completion as an example of this situation. Proofs of the statements in this subsection appear, e.g., in [**KashiwaraShapira:Cate**

Definition 8.24. Let C be a small category. The *Ind-completion* $\mathrm{Ind}(C)$ is defined to be the full subcategory of $\mathrm{PSh}(C)$ consisting of those presheaves $F : C \to \mathrm{Set}$ such that there is a *filtered* diagram

$$K : I \to C$$

(i.e., I is a filtered category) and an isomorphism

$$F \cong \operatorname{colim} y(K(i)).$$

Remark 8.25.

- The important point is to insist that K is filtered. Indeed, the formula

$$F \cong \int^i \mathrm{Hom}_I(-, i) \otimes F(i) = \int^i y(i) \otimes F(i)$$

(Lemma 8.14) shows that we can write any presheaf F as a colimit (namely, the coend above, which is a colimit involving of a diagram involving the terms $y(i) \otimes F(i) = \coprod_{* \in F(i)} y(i)$) of a diagram involving only the representable presheaves $y(i)$.

- By definition, we can think of the objects in $\mathrm{Ind}(C)$ as being formal filtered colimits:

$$F = \text{“ } \operatorname{colim} K_i \text{ ”}.$$

This is formal in the sense that the category C need not have such a colimit (and even if it does, the Yoneda embedding need not preserve it).

Remark 8.26. We compute the set of morphisms $\mathrm{Hom}_{\mathrm{Ind}(C)}(F, F')$. Suppose $F = \operatorname{colim} y(K(i))$ and $F' = \operatorname{colim} y(K'(i'))$ for $K : I \to C$ and $K' : I' \to C$ two filtered diagrams.

$$\begin{aligned}
\mathrm{Hom}(F, F') &= \mathrm{Hom}_{\mathrm{PSh}(C)} \left(\operatorname{colim}_i y(K(i)), \operatorname{colim}_{i'} y(K'(i'))\right) \\
&= \lim_i \mathrm{Hom}_{\mathrm{PSh}(C)} \left(y(K(i)), \operatorname{colim}_{i'} y(K'(i'))\right) \\
&= \lim_i \left(\operatorname{colim}_{i'} y(K'(i'))\right)(K(i)) \\
&\overset{*}{=} \lim_i \operatorname{colim}_{i'} \left(y(K'(i'))\right)(K(i)) \\
&= \lim_i \operatorname{colim}_{i'} \mathrm{Hom}_C(K'(i'), K(i)).
\end{aligned}$$

At $*$, we have used that colimits in functor categories are computed pointwise (Lemma 4.34). Thus, given a presentation of F and F' as above, a morphism $f : F \to F'$ is a collection of maps, for each i,

$$K(i) \to K'(i'_i),$$

where $i'_i \in I'$ is an index (depending on i). These maps are supposed to be compatible as i varies (in order to be an element in $\lim_i$, as opposed to $\prod_i$).

Theorem 8.27. Let C be a small category.
(1) The category $\mathrm{Ind}(C)$ has all filtered colimits.
(2) For any category D that has all filtered colimits, the composite

$$\mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(C), D) \subset \mathrm{Fun}(\mathrm{Ind}(C), D) \xrightarrow{y^*} \mathrm{Fun}(C, D)$$

is an equivalence. Here the left hand category is the full subcategory of the functor category consisting of functors preserving filtered colimits. An inverse is given by left Kan extensions; it is also referred to as the *Ind-extension* of a given functor $F : C \to D$.
(3) The Yoneda embedding $y : C \subset \mathrm{PSh}(C)$ takes values in the full subcategory $\mathrm{Ind}(C) \subset \mathrm{PSh}(C)$. For any $c \in C$, $y(c)$ is a *compact* object in $\mathrm{Ind}(C)$.

Proof. We only indicate some bits for (2): $y(C) \subset \mathrm{Ind}(C)$ by definition. The inclusion

$$\mathrm{Ind}(C) \subset \widehat{C}$$

preserves filtered colimits (note that the inclusion $C \to \mathrm{Ind}(C)$ need not preserve them; compare this with Exercise 4.16). This follows from the second statement, applied to $D = \widehat{C}$. Then, the functor

$$\mathrm{Hom}_{\mathrm{Ind}(C)}(y(c), -) = \mathrm{Hom}_{\widehat{C}}(y(c), -) = -(c)$$

preserves filtered colimits, since the right hand functor in fact preserves all colimits (Lemma 4.34). $\square$

In order to understand the Ind-completion more concretely, the following statement is useful:

Corollary 8.28. Let $F : C \to D$ be a functor, with D admitting all filtered colimits, and $\overline{F} : \mathrm{Ind}(C) \to D$ the Ind-extension of F.
(1) $\overline{F}$ is fully faithful if F is fully faithful and if $F(c)$ is a compact object (in D) for each $c \in C$.

(2) $\overline{F}$ is an equivalence if, in addition, every object $d \in D$ can be presented as a filtered colimit $d = \operatorname{colim} F(c_i)$, for an appropriate filtered diagram $I \to C$.

Proof. (1) follows quickly from the computation of Hom-sets in $\mathrm{Ind}(C)$. □

Example 8.29. We have

$$\mathrm{Set} = \mathrm{Ind}(\mathrm{FinSet}), \mathrm{Vect} = \mathrm{Ind}(\mathrm{Vect}^{fd})$$

(the category Vect^{fd} denotes the full subcategory of finite-dimensional vector spaces). Indeed, finite-dimensional vector spaces are compact objects in Vect (Example 4.48). In addition, each vector space V is a filtered colimit of finite-dimensional ones:

$$V = \operatorname{colim}_{W \subset V, \dim W < \infty} W.$$

8.5 Exercises

Exercise 8.1. Let C be a category, $X \in C$ an object and $f : X \to X$ an endomorphism. We regard the pair (X, f) as a functor $B\mathbf{N} \to C$ (cf. Exercise 2.1) by sending $\star \mapsto X$ and $\mathrm{Hom}(\star, \star) = \mathbf{N} \ni n \mapsto f^{\circ n}$.

Show that the left, resp. right Kan extensions along the natural functor $i : B\mathbf{N} \to B\mathbf{Z}$ are given by

$$\mathrm{Lan}_i(X, f) = \operatorname{colim}(X \xrightarrow{f} X \xrightarrow{f} \dots),$$

$$\mathrm{Ran}_i(X, f) = \lim(\cdots \xrightarrow{f} X \xrightarrow{f} X).$$

Exercise 8.2. Let C be a small category. What is the left Kan extension in this diagram:

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, "y"] & \widehat{C} \\ \widehat{C} \arrow[ur, "\mathrm{Lan}_y y = ?"'] & \end{tikzcd}$$

Exercise 8.3. Let $f : C \to C'$ be a functor between small categories. Consider the precomposition-with-f-functor

$$f^* : \widehat{C'} \to \widehat{C}$$

as well as the Kan extension

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "y := y_C"] & C' \arrow[d, "y' := y_{C'}"] \\ \widehat{C} \arrow[r, "\mathrm{Lan}_y(y' f)"] & \widehat{C'}. \end{tikzcd}$$

Show that

$$\mathrm{Lan}_y(y' f) : \widehat{C} \rightleftarrows \widehat{C'} : f^*$$

are adjoint.

References

[Awo10] Steve Awodey. *Category theory*. Second. Vol. 52. Oxford Logic Guides. Oxford University Press, Oxford, 2010, pp. xvi+311. ISBN: 978-0-19-923718-0.

[Bor94] Francis Borceux. *Handbook of categorical algebra. 1*. Vol. 50. Encyclopedia of Mathematics and its Applications. Basic category theory. Cambridge University Press, Cambridge, 1994, pp. xvi+345. ISBN: 0-521-44178-1.

[Bra15] M. Brandenburg. *Einführung in die Kategorientheorie: Mit ausführlichen Erklärungen und zahlreichen Beispielen*. Springer Berlin Heidelberg, 2015. ISBN: 9783662470688.

[Fre70] Peter Freyd. "Homotopy is not concrete". In: *The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970)*. Lecture Notes in Mathematics, Vol. 168. Springer, Berlin, 1970, pp. 25–34.

[Lei14] Tom Leinster. *Basic category theory*. Vol. 143. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2014, pp. viii+183. ISBN: 978-1-107-04424-1. DOI: 10.1017/CBO9781107360068. URL: https://doi.org/10.1017/CBO9781107360068.

[Mac98] Saunders Mac Lane. *Categories for the working mathematician*. Second. Vol. 5. Graduate Texts in Mathematics. New York: Springer-Verlag, 1998, pp. xii+314. ISBN: 0-387-98403-8.

[Rie17] E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications, 2017. ISBN: 9780486820804. URL: http://www.math.jhu.edu/~eriehl/context.pdf.